

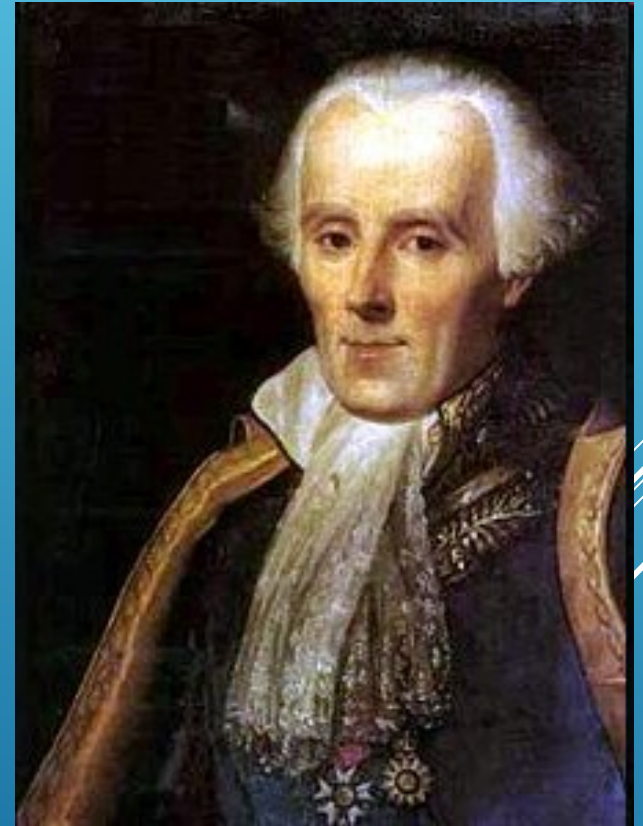
LAPLACE TRANSFORM



THE FRENCH NEWTON

PIERRE-SIMON LAPLACE

- ▮ Developed mathematics in astronomy, physics, and statistics
- ▮ Began work in calculus which led to the Laplace Transform
- ▮ Focused later on celestial mechanics
- ▮ One of the first scientists to suggest the existence of black holes



HISTORY OF THE TRANSFORM

- Euler began looking at integrals as solutions to differential equations in the mid 1700's:

$$z = \int X(x)e^{ax} dx$$

- Lagrange took this a step further while working on probability density functions and looked at forms of the following equation:

$$\int X(x)e^{-ax} a^x dx,$$

$$z(x; a) = \int_0^x e^{at} X(t) dt,$$

- Finally, in 1785, Laplace began using a transformation to solve equations of finite differences which eventually lead to the current transform

$$S = Ay_s + B \Delta y_s + C \Delta^2 y_s + \dots,$$

$$y_s = \int e^{-sx} \phi(x) dx,$$

□ The Laplace transform is a linear operator that switched a function $f(t)$ to $F(s)$.

□ Specifically:

where:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt.$$

□ Go from time argument with real input to a complex angular frequency input which is complex. $s = \sigma + i\omega$.

DEFINITION

- There are two governing factors that determine whether Laplace transforms can be used:
 - $f(t)$ must be at least piecewise continuous for $t \geq 0$
 - $|f(t)| \leq Me^{\gamma t}$ where M and γ are constants

RESTRICTIONS



- Since the general form of the Laplace transform is:

it makes sense that $f(t)$ must be at least piecewise continuous for $t \geq 0$.

- If $f(t)$ were very nasty, the integral would not be computable.

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt.$$

CONTINUITY

- This criterion also follows directly from the general definition:
- If $f(t)$ is not bounded by $Me^{\gamma t}$ then the integral will not converge.

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt.$$

BOUNDEDNESS

LAPLACE TRANSFORM THEORY

- General Theory
- Example
- Convergence

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt = \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{-st} f(t) dt$$

$$f(t) \equiv 1$$

$$\begin{aligned} \mathcal{L}(f(t)) &= \int_0^{\infty} e^{-st} 1 dt = \lim_{\tau \rightarrow \infty} \left(\frac{e^{-st}}{-s} \Big|_0^{\tau} \right) \\ &= \lim_{\tau \rightarrow \infty} \left(\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) = \frac{1}{s} \end{aligned}$$

$$f(t) \equiv e^{t^2}$$

$$\mathcal{L}(f(t)) = \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{-st} e^{t^2} dt = \lim_{\tau \rightarrow \infty} \int_0^{\tau} e^{t^2 - st} dt = \infty$$

LAPLACE TRANSFORMS

- Some Laplace Transforms
- Wide variety of function can be transformed
- Inverse Transform

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

- Often requires partial fractions or other manipulation to find a form that is easy to apply the inverse

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > a $
8. $\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > a $
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
11. $t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$

LAPLACE TRANSFORM FOR ODES

- Equation with initial conditions
- Laplace transform is linear
- Apply derivative formula
- Rearrange
- Take the inverse

$$\frac{d^2y}{dt^2} + y = 1, \quad y(0) = y'(0) = 0$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(1)$$

$$s^2 \mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{1}{s}$$

$$\mathcal{L}(y) = \frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$y = 1 - \cos t$$

Laplace Transform in PDEs

Laplace transform in two variables (always taken with respect to time variable, t):

Inverse laplace of a 2 dimensional PDE:

Can be used for any dimension PDE:

The Transform reduces dimension by “1”:

- ODEs reduce to algebraic equations
- PDEs reduce to either an ODE (if original equation dimension 2) or another PDE (if original equation dimension >2)

$$\mathcal{L}\{u(x,t)\} = U(x,s) = \int_0^{\infty} e^{-st} \frac{du}{dt} dt$$

$$\mathcal{L}^{-1}\{U(x,s)\} = u(x,t)$$

$$\mathcal{L}\{u(x,y,z,t)\} = U(x,y,z,s)$$

Consider the case where:

$$u_x + u_t = t \quad \text{with } u(x,0)=0 \text{ and } u(0,t)=t^2 \quad \text{and}$$

Taking the Laplace of the initial equation leaves $U_x + U = 1/s^2$ (note that the partials with respect to “x” do not disappear) with boundary condition $U(0,s) = 2/s^3$

Solving this as an ODE of variable x, $U(x,s) = c(s)e^{-x} + 1/s^2$

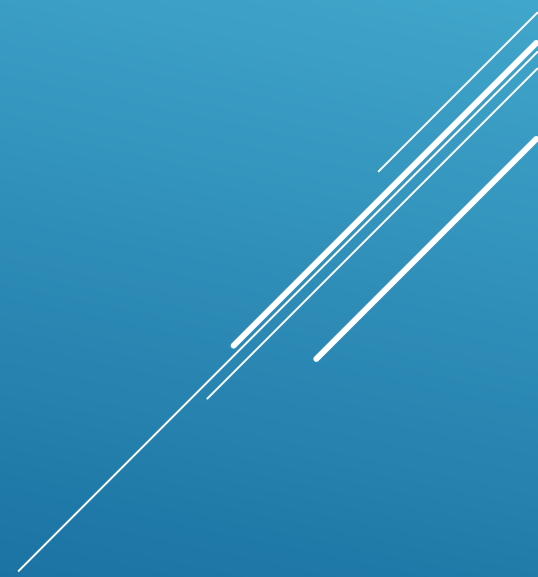
Plugging in B.C., $2/s^3 = c(s) + 1/s^2$ so $c(s) = 2/s^3 - 1/s^2$

$$U(x,s) = (2/s^3 - 1/s^2) e^{-x} + 1/s^2$$

Now, we can use the inverse Laplace Transform with respect to s to find

$$u(x,t) = t^2 e^{-x} - t e^{-x} + t$$

EXAMPLE SOLUTIONS



DIFFUSION EQUATION

$$u_t = ku_{xx} \text{ in } (0,l)$$

Initial Conditions:

$$u(0,t) = u(l,t) = 1, \quad u(x,0) = 1 + \sin(\pi x/l)$$

Using $af(t) + bg(t) \Rightarrow aF(s) + bG(s)$

and $df/dt \Rightarrow sF(s) - f(0)$

and noting that the partials with respect to x commute with the transforms with respect to t , the Laplace transform $U(x,s)$ satisfies

$$sU(x,s) - u(x,0) = kU_{xx}(x,s)$$

With $e^{at} \Rightarrow 1/(s-a)$ and $a=0$,

the boundary conditions become $U(0,s) = U(l,s) = 1/s$.

So we have an ODE in the variable x together with some boundary conditions.
The solution is then:

$$U(x,s) = 1/s + (1/(s+k\pi^2/l^2))\sin(\pi x/l)$$

Therefore, when we invert the transform, using the Laplace table:

$$u(x,t) = 1 + e^{-k\pi^2 t/l^2} \sin(\pi x/l)$$

WAVE EQUATION

$$u_{tt} = c^2 u_{xx} \text{ in } 0 < x < \infty$$

Initial Conditions:

$$u(0,t) = f(t), \quad u(x,0) = u_t(x,0) = 0$$

For $x \rightarrow \infty$, we assume that $u(x,t) \rightarrow 0$. Because the initial conditions vanish, the Laplace transform satisfies

$$s^2 U = c^2 U_{xx}$$

$$U(0,s) = F(s)$$

Solving this ODE, we get

$$U(x,s) = a(s)e^{-sx/c} + b(s)e^{sx/c}$$

Where $a(s)$ and $b(s)$ are to be determined.

From the assumed property of u , we expect that $U(x,s) \rightarrow 0$ as $x \rightarrow \infty$.

Therefore, $b(s) = 0$. Hence, $U(x,s) = F(s) e^{-sx/c}$. Now we use

$$\mathcal{H}(t-b)f(t-b) \rightarrow e^{-bs}F(s)$$

To get

$$u(x,t) = H(t - x/c)f(t - x/c).$$

Real-Life Applications

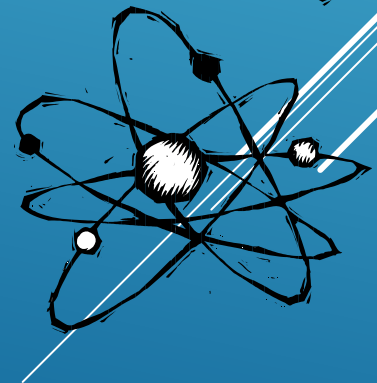
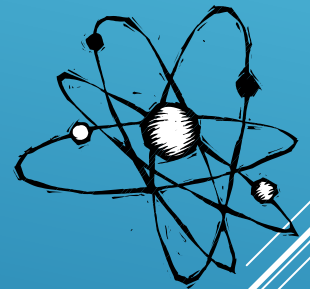
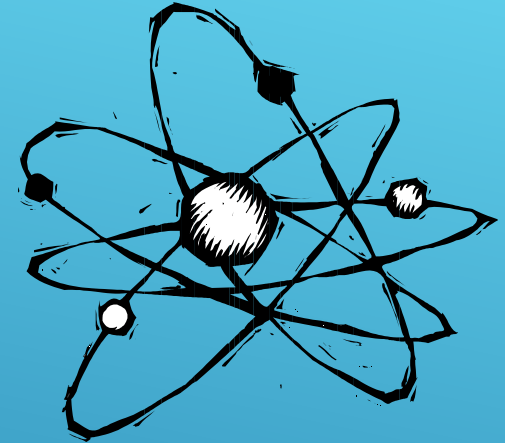
- Semiconductor mobility
- Call completion in wireless networks
- Vehicle vibrations on compressed rails
- Behavior of magnetic and electric fields above the atmosphere



Ex. Semiconductor Mobility

● Motivation

- semiconductors are commonly made with superlattices having layers of differing compositions
- need to determine properties of carriers in each layer
 - concentration of electrons and holes
 - mobility of electrons and holes
- conductivity tensor can be related to Laplace transform of electron and hole densities



Notation

- R = ratio of induced electric field to the product of the current density and the applied magnetic field
- ρ = electrical resistance
- H = magnetic field
- J = current density
- E = applied electric field
- n = concentration of electrons
- u = mobility

$$E_x = \rho J_x - RHJ_y$$

$$E_y = RHJ_x + \rho J_y$$

Equation Manipulation

$$\rho = \frac{1}{ne\mu}$$

and

$$R = -\frac{1}{ne}$$

$$\sigma_{xx} = \frac{ne\mu}{[1 + (\mu H)]^2}$$

$$\sigma_{xy} = \frac{-ne\mu^2 H}{[1 + (\mu H)]^2}$$



$$J_x = \sigma_{xx}E_x + \sigma_{xy}E_y,$$

$$J_y = -\sigma_{xy}E_x + \sigma_{xx}E_y,$$

Assuming a continuous mobility distribution and that $S^+(\mu) = k^+ e \mu p_\mu$, $S^-(\mu) = k^- e \mu n_\mu$, it follows:

$$\sigma_{xx} = \sum_{\mu} \frac{(n_{\mu} + p_{\mu}) e \mu}{[1 + (\mu H)]^2},$$

$$\sigma_{xy} = \sum_{\mu} \frac{(p_{\mu} - n_{\mu}) (e \mu^2 H)}{[1 + (\mu H)]^2}$$

$$\sigma_{xx}(H) = \int_0^{\infty} \frac{S^+(\mu) + S^-(\mu)}{1 + (\mu H)^2} d\mu,$$

$$\sigma_{xy}(H) = \int_0^{\infty} \frac{[S^+(\mu) - S^-(\mu)] \mu H}{1 + (\mu H)^2} d\mu.$$

Applying the Laplace Transform

$$\int_0^{\infty} e^{-yt} \sin(xt) dt = \frac{x}{x^2 + y^2},$$

$$\int_0^{\infty} e^{-yt} \cos(xt) dt = \frac{y}{x^2 + y^2}$$



$$\begin{aligned} H\sigma_{xx} &= \int_0^{\infty} e^{-yt} \left[\int_0^{\infty} (S^+ + S^-) \cos(xt) dx \right] dt \\ &= \int_0^{\infty} e^{-yt} \left[\int_0^{\infty} ex(p_x + n_x) \cos(xt) dx \right] dt. \end{aligned}$$

$$\begin{aligned} H\sigma_{xy} &= \int_0^{\infty} e^{-yt} \left[\int_0^{\infty} (S^+ - S^-) \sin(xt) dx \right] dt \\ &= \int_0^{\infty} e^{-yt} \left[\int_0^{\infty} ex(p_x - n_x) \sin(xt) dx \right] dt. \end{aligned}$$

Source

Johnson, William B. Transform method for semiconductor mobility, Journal of Applied Physics 99 (2006).

