The Inverse Laplace Transforms

Def: Inverse Laplace Transform:

If $f(P)$ is the L·T of the function $F(t)$ i.e
$L\{F(t)\} = f(P)$, then $F(t)$ is called the Inverse Laplace Transform of the function $f(P)$ and it is written as

$$F(t) = L^{-1}\{f(P)\}$$, where L^{-1} is called the Inverse Laplace Transform operator.

Null Function:

If $N(t)$ is a real valued function such that

$\int_0^t N(t)\,dt = 0 \Rightarrow N(t)$ is called a null function.

Theorem: Linear Property:

If $f_1(P)$ and $f_2(P)$ be the Laplace transforms of the functions $F_1(t)$ and $F_2(t)$ respectively. Then for any constants c_1 and c_2 we have

$$L^{-1}\{c_1 f_1(P) + c_2 f_2(P)\} = c_1 L^{-1}\{f_1(P)\} + c_2 L^{-1}\{f_2(P)\}$$

Proof: Given that:

$L\{F_1(t)\} = f_1(P)$ and $L\{F_2(t)\} = f_2(P)$

$\Rightarrow F_1(t) = L^{-1}\{f_1(P)\}$ and $F_2(t) = L^{-1}\{f_2(P)\}$

Take c_1, c_2 are arbitrary constants

Then

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = L\{c_1 F_1(t)\} + L\{c_2 F_2(t)\}$$
$$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$
$$= c_1 f_1(P) + c_2 f_2(P)$$

$$\therefore L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 f_1(P) + c_2 f_2(P)$$
$$\Rightarrow c_1 F_1(t) + c_2 F_2(t) = L^{-1}\{c_1 f_1(P) + c_2 f_2(P)\}$$
$$\Rightarrow c_1 L^{-1}\{f_1(P)\} + c_2 L^{-1}\{f_2(P)\} = L^{-1}\{c_1 f_1(P) + c_2 f_2(P)\}$$
$$\therefore L^{-1}\{c_1 f_1(P) + c_2 f_2(P)\} = c_1 L^{-1}\{f_1(P)\} + c_2 L^{-1}\{f_2(P)\},$$

c_1, c_2 are ar constants.

Hence Proved

—x—

Some standard Results:

1. $L(1) = \frac{1}{P}, \; P > 0 \Rightarrow L^{-1}\{\frac{1}{P}\} = 1$

2. $L\{t^n\} = \frac{n!}{P^{n+1}} \Rightarrow L^{-1}\{\frac{n!}{P^{n+1}}\} = t^n$

$$\Rightarrow L^{-1}\{\frac{1}{P^{n+1}}\} = \frac{t^n}{n!}$$

3. $L(e^{at}) = \frac{1}{p-a} \Rightarrow L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

4. $L(e^{-at}) = \frac{1}{p+a} \Rightarrow L^{-1}\left\{\frac{1}{p+a}\right\} = e^{-at}$

5. $L\{\sin at\} = \frac{a}{p^2+a^2} \Rightarrow L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$

6. $L\{\cos at\} = \frac{p}{p^2+a^2} \Rightarrow L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

7. $L\{\sinh at\} = \frac{a}{p^2-a^2} \Rightarrow L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}\sinh at$

8. $L\{\cosh at\} = \frac{p}{p^2-a^2} \Rightarrow L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

1. Prove that $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}$

Sol: we know that:

$$L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}, \quad p>0$$

$$\Rightarrow t^n = L^{-1}\left\{\frac{\Gamma(n+1)}{p^{n+1}}\right\} = \Gamma(n+1)\, L^{-1}\left\{\frac{1}{p^{n+1}}\right\}$$

$$= n!\, L^{-1}\left\{\frac{1}{p^{n+1}}\right\}$$

$$\therefore t^n = n!\, L^{-1}\left\{\frac{1}{p^{n+1}}\right\} \Rightarrow \frac{t^n}{n!} = L^{-1}\left\{\frac{1}{p^{n+1}}\right\}$$

$$\boxed{\therefore L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}}$$ Imp

2. Find $L^{-1}\left\{\frac{1}{p^4}\right\}$, $L^{-1}\left\{\frac{1}{p^2+4}\right\}$, $L^{-1}\left\{\frac{4}{p-2}\right\}$, $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$

Sol: i) we know that

$$L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}$$

put $n=3$

$$L^{-1}\left\{\frac{1}{p^4}\right\} = \frac{t^3}{3!} = \frac{t^3}{6}$$

ii) we know that : $\boxed{L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at}$

put $a=2$

$$L^{-1}\left\{\frac{1}{p^2+4}\right\} = \frac{1}{2}\sin 2t$$

iii) we know that : $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

$$\Rightarrow L^{-1}\left\{\frac{1}{p-2}\right\} = e^{2t}$$

$$\therefore L^{-1}\left\{\frac{4}{p-2}\right\} = 4e^{2t}$$

iv) we know that $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$

put $n = -\frac{1}{2}$

$$L^{-1}\left\{\frac{1}{p^{-\frac{1}{2}+1}}\right\} = \frac{t^{-\frac{1}{2}}}{\Gamma(-\frac{1}{2}+1)}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^{1/2}}\right\} = \frac{t^{-1/2}}{\Gamma(\frac{1}{2})}$$

$$\Rightarrow L^{-1}\left(\frac{1}{\sqrt{p}}\right) = \frac{1}{t^{1/2}\,\cancel{\tfrac{1}{2}}\sqrt{\pi}} = \frac{1}{\sqrt{t}\sqrt{\pi}} = \frac{1}{\sqrt{t\pi}}$$

$$\therefore L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = \frac{1}{\sqrt{t\pi}}$$

3. Find $L^{-1}\left\{\frac{1}{p^{7/2}}\right\}$

Sol: we know that $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$

put $n = \frac{5}{2} \Rightarrow n+1 = \frac{7}{2}$

$$L^{-1}\left\{\frac{1}{p^{5/2+1}}\right\} = \frac{t^{5/2}}{\Gamma(\frac{5}{2}+1)}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^{7/2}}\right\} = \frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{8t^{5/2}}{15\sqrt{\pi}}$$

$$\boxed{\therefore L^{-1}\left\{\frac{1}{p^{7/2}}\right\} = \frac{8t^{5/2}}{15\sqrt{\pi}}}$$

4. Find $L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\}$

Sol:

$$L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\} = L^{-1}\left\{\frac{p}{p^2+2}\right\} + 6L^{-1}\left\{\frac{p}{p^2-16}\right\} + 3L^{-1}\left\{\frac{1}{p-3}\right\}$$

$$= L^{-1}\left\{\frac{p}{p^2+(\sqrt{2})^2}\right\} + 6L^{-1}\left\{\frac{p}{p^2-4^2}\right\} + 3e^{3t}$$

$$= \cos\sqrt{2}t + 6\cosh(4t) + 3e^{2t}$$

$$\left\{ L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at,\ L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at,\ L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at} \right\}$$

5. Find $L^{-1}\left\{\frac{6}{2p-3}+\frac{3+4p}{9p^2-16}+\frac{8-6p}{16p^2+9}\right\}$

Sol:

$$L^{-1}\left\{\frac{6}{2p-3}\right\} = 6\,L^{-1}\left\{\frac{1}{2p-3}\right\} = \frac{6}{2}\,L^{-1}\left\{\frac{1}{p-\frac{3}{2}}\right\}$$

$$= 3\,e^{\frac{3}{2}t}$$

$$L^{-1}\left\{\frac{3+4p}{9p^2-16}\right\} = L^{-1}\left\{\frac{3}{9p^2-16}+\frac{4p}{9p^2-16}\right\}$$

$$= \frac{3}{9}\,L^{-1}\left\{\frac{1}{p^2-\frac{16}{9}}\right\} + \frac{4}{9}\,L^{-1}\left\{\frac{p}{p^2-\frac{16}{9}}\right\}$$

$$= \frac{1}{3}\,L^{-1}\left\{\frac{1}{p^2-\left(\frac{4}{3}\right)^2}\right\} + \frac{4}{9}\,L^{-1}\left\{\frac{p}{p^2-\left(\frac{4}{3}\right)^2}\right\}$$

$$= \frac{1}{3}\times\frac{3}{4}\,\sinh\left(\frac{4}{3}\right)t + \frac{4}{9}\cosh\left(\frac{4}{3}\right)t$$

$$= \frac{1}{4}\sinh\left(\frac{4}{3}\right)t + \frac{4}{9}\cosh\left(\frac{4}{3}\right)t$$

$$L^{-1}\left\{\frac{8-6p}{16p^2+9}\right\} = L^{-1}\left\{\frac{8}{16p^2+9}-\frac{6p}{16p^2+9}\right\}$$

$$= L^{-1}\left\{\frac{8}{16p^2+9}\right\} - L^{-1}\left\{\frac{6p}{16p^2+9}\right\}$$

$$= \frac{8}{16}\,L^{-1}\left\{\frac{1}{p^2+\frac{9}{16}}\right\} - \frac{6}{16}\,L^{-1}\left\{\frac{p}{p^2+\frac{9}{16}}\right\}$$

$$= \frac{1}{2}\,L^{-1}\left\{\frac{1}{p^2+\left(\frac{3}{4}\right)^2}\right\} - \frac{3}{8}\,L^{-1}\left\{\frac{p}{p^2+\left(\frac{3}{4}\right)^2}\right\}$$

$$= \frac{1}{\cancel{2}}\times\frac{\cancel{4}^{2}}{3}\sin\left(\frac{3}{4}\right)t - \frac{3}{8}\cos\left(\frac{3}{4}\right)t$$

Now

$$L^{-1}\left\{\frac{6}{2p-3}-\frac{3+4p}{9p^2-16}+\frac{8-6p}{16p^2+9}\right\} = L^{-1}\left\{\frac{6}{2p-3}\right\} - L^{-1}\left\{\frac{3+4p}{9p^2-16}\right\} + L^{-1}\left\{\frac{8-6p}{16p^2+9}\right\}$$

$$= 3e^{\frac{3}{2}t} + \frac{1}{4}\sinh\left(\frac{4}{3}\right)t + \frac{4}{9}\cosh\left(\frac{4}{3}\right)t + \frac{2}{3}\sin\left(\frac{3}{4}\right)t - \frac{3}{8}\cos\left(\frac{3}{4}\right)t$$

= Ans

6. Find $L^{-1}\left\{\frac{3}{p^2-3}+\frac{3p+2}{p^3}-\frac{3p-27}{p^2-9}+\frac{6-30\sqrt{p}}{p^4}\right\}$

Sol:

$$L^{-1}\left\{\frac{3}{p^2-3}\right\} = 3\,L^{-1}\left\{\frac{1}{p^2-(\sqrt{3})^2}\right\} = 3\cdot\frac{1}{\sqrt{3}}\sinh\sqrt{3}t$$

$$= \sqrt{3}\sinh\sqrt{3}t \longrightarrow (1)$$

$$L^{-1}\left\{\frac{3p+2}{p^3}\right\} = L^{-1}\left\{\frac{3p}{p^3}+\frac{2}{p^3}\right\} = 3L^{-1}\left\{\frac{1}{p^2}\right\}+2L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$= 3\cdot\frac{t}{1!}+2\frac{t^2}{2!} = 3t+t^2 \longrightarrow (2)$$

$$L^{-1}\left\{\frac{3p-27}{p^2-9}\right\} = L^{-1}\left\{\frac{3p}{p^2-9}\right\} - L^{-1}\left\{\frac{27}{p^2-9}\right\}$$

$$= 3L^{-1}\left\{\frac{p}{p^2-9}\right\} - 27L^{-1}\left\{\frac{1}{p^2-3^2}\right\}$$

$$= 3\cosh 3t - 27\cdot\frac{1}{3}\sinh 3t$$

$$= 3\cosh 3t - 9\sinh 3t \longrightarrow (3)$$

$$L^{-1}\left\{\frac{6-30\sqrt{p}}{p^4}\right\} = L^{-1}\left\{\frac{6}{p^4}-\frac{30\sqrt{p}}{p^4}\right\}$$

$$= 6L^{-1}\left\{\frac{1}{p^4}\right\} - 30\,L^{-1}\left\{\frac{1}{p^{5/2+1}}\right\}$$

$$= 6\frac{t^3}{3!} - 30\times\frac{8}{15}\frac{t^{5/2}}{\pi}$$

$$= t^3 - \frac{16\,t^{5/2}}{\sqrt{\pi}} \longrightarrow (4)$$

Now

$$L^{-1}\left\{\frac{3}{p^2-3}+\frac{3p+2}{p^3}-\frac{3p-27}{p^2-9}+\frac{6-30\sqrt{p}}{p^4}\right\} =$$

$$L^{-1}\left\{\frac{3}{p^2-3}\right\} + L^{-1}\left\{\frac{3p+2}{p^3}\right\} - L^{-1}\left\{\frac{3p-27}{p^2-9}\right\} + L^{-1}\left\{\frac{6-30\sqrt{p}}{p^4}\right\}$$

$\cancel{= 3t + t^2 + 3\cosh 3t + 9\sinh 3t}$

$$= \sqrt{3}\sinh\sqrt{3}t + 3t + t^2 - 3\cosh 3t + 9\sinh 3t + t^3 - \frac{16t^{5/2}}{\sqrt{\pi}}$$

=Ans.

7. Show that: $L^{-1}\left\{\frac{5}{p^2} + \left(\frac{\sqrt{p}-1}{p}\right)^2 - \frac{1}{3p+2}\right\} = 1 + 6t - 4\sqrt{\frac{t}{\pi}} - \frac{7}{3}e^{-2t/3}$

Sol:

$$L^{-1}\left\{\frac{5}{p^2} + \left(\frac{\sqrt{p}-1}{p}\right)^2 - \frac{1}{3p+2}\right\} = L^{-1}\left\{\frac{5}{p^2}\right\} + L^{-1}\left\{\frac{p+1-2\sqrt{p}}{p^2}\right\} - L^{-1}\left\{\frac{7}{3p+2}\right\}$$

$$= 5L^{-1}\left\{\frac{1}{p^2}\right\} + L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p^2}\right\} - 2L^{-1}\left\{\frac{1}{p^{3/2}}\right\} - \frac{7}{3}L^{-1}\left\{\frac{1}{p+\frac{3}{2}}\right\}$$

$$= 5t + 1 + t - 2\frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} - \frac{7}{3}e^{-2/3t}$$

$$= 6t + 1 - 4\sqrt{\frac{t}{\pi}} - \frac{7}{3}e^{-2/3t} = \text{Ans.}$$

8. Find the inverse Laplace Transform of the following functions:

i) $\frac{1}{p^5}$ ii) $\frac{1}{2p-5}$ iii) $\frac{2p+1}{p(p+1)}$ iv) $\frac{2p-5}{p^2-9}$

v) $\frac{3p-8}{4p^2-25}$ vi) $\frac{3p-2}{p^{5/2}} - \frac{7}{3p+2}$

vii) $\frac{3(p^2-1)^2}{2p^5} + \frac{4p-18}{9-p^2} + \frac{(p+1)(2-\sqrt{p})}{p^{5/2}}$

First Shifting Theorem:

If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f(p-a)\} = e^{at}F(t)$

Proof: we know that:

$$f(p) = \int_0^\infty e^{-pt}F(t)\,dt, \text{ then}$$

$$f(p-a) = \int_0^\infty e^{-(p-a)t}F(t)\,dt$$

$$= \int_0^\infty e^{-pt}e^{at}F(t)\,dt = \int_0^\infty e^{-pt}\{e^{at}F(t)\}\,dt$$

$$= L\{e^{at}F(t)\}$$

$$\therefore f(p-a) = L\{e^{at}F(t)\}$$

i.e $L^{-1}\{f(p-a)\} = e^{at}F(t)$

Hence proved

Second Shifting Theorem:

If $L^{-1}\{f(P)\} = F(t)$, then $L^{-1}\{e^{-aP} f(P)\} = G(t)$, where

$$G(t) = F(t-a) \text{ (or) } G(t) = \begin{cases} F(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$$

Proof:

Let $L^{-1}\{f(P)\} = F(t)$, we have

$$f(P) = \int_0^\infty e^{-Pt} F(t)\, dt$$

$$\Rightarrow e^{-aP} f(P) = \int_0^\infty e^{-Pt} e^{-aP} F(t)\,dt$$

$$= \int_0^\infty e^{-(Pt+aP)} F(t)\,dt$$

$$= \int_0^\infty e^{-P(t+a)} F(t)\,dt$$

put $t+a=x$
$\Rightarrow dt = dx$

$$= \int_0^\infty e^{-Px} F(x-a)\,dx$$

$$= \int_0^a e^{-Px} F(x-a)\,dx + \int_a^\infty e^{-Px} F(x-a)\,dx$$

$$= \int_0^a e^{-Px} \cdot 0 \cdot dx + \int_a^\infty e^{-Px} F(x-a)\,dx$$

$$= \int_a^\infty e^{-Px} F(x-a)\,dx = \int_a^\infty e^{-Px} G(t)\,dt$$

$$= L\{G(t)\}$$

$$\therefore e^{-aP} f(P) = L\{G(t)\}$$

i.e $L^{-1}\{e^{-aP} f(P)\} = G(t)$

Hence Proved

Change of Scale Property:

If $L^{-1}\{f(P)\} = F(t)$, then $L^{-1}\{f(aP)\} = \frac{1}{a} F\left(\frac{t}{a}\right)$

Proof: Let $L^{-1}\{f(P)\} = F(t)$, we have

$$f(P) = \int_0^\infty e^{-Pt} F(t)\,dt$$

Now $f(ap) = \int_0^\infty e^{-apt} F(t)\,dt$

put $t = \frac{x}{a}$
$\Rightarrow at = x$
$\Rightarrow a\,dt = dx$
$\Rightarrow dt = \frac{dx}{a}$

$= \int_0^\infty e^{-p(at)} F(t)\,dt$

$= \int_0^\infty e^{-px} F\left(\frac{x}{a}\right) \frac{dx}{a}$

$= \int_0^\infty e^{-px} \left\{ \frac{1}{a} F\left(\frac{x}{a}\right) \right\} dx$

$= L\left\{ \frac{1}{a} F\left(\frac{x}{a}\right) \right\}$

$\therefore \quad f(ap) = L\left\{ \frac{1}{a} F\left(\frac{x}{a}\right) \right\}$

i.e $L^{-1}\{f(ap)\} = \frac{1}{a} F\left(\frac{x}{a}\right)$

$\therefore L^{-1}\{f(ap)\} = \frac{1}{a} F\left(\frac{t}{a}\right)$ Ans.

Use of Partial Fractions:

If $f(p)$ is of the form $\frac{g(p)}{h(p)}$, where g, h are polynomials in p, then break $f(p)$ into partial fractions and manipulate them term by term.

1. Find $L^{-1}\left\{\frac{1}{(p+a)^n}\right\}$

Sol: $L^{-1}\left\{\frac{1}{(p+a)^n}\right\} = e^{-at} L^{-1}\left\{\frac{1}{p^n}\right\} = e^{-at} \frac{t^{n-1}}{(n-1)!}$

$\therefore L^{-1}\left\{\frac{1}{(p+a)^n}\right\} = e^{-at} \frac{t^{n-1}}{(n-1)!}$

2. Find $L^{-1}\left\{\frac{1}{(p-a)^n}\right\} = e^{at} L^{-1}\left\{\frac{1}{p^n}\right\} = e^{at} \frac{t^{n-1}}{(n-1)!}$

3. Find $L^{-1}\left\{\frac{1}{(p-3)^2}\right\} = e^{3t} L^{-1}\left\{\frac{1}{p^2}\right\} = e^{3t} \cdot t$

4. $L^{-1}\left\{\frac{1}{(p+2)^5}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{p^5}\right\} = e^{-2t} \frac{t^4}{4!}$

5. Find $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\}$

$$L^{-1}\left\{\frac{1}{p^2-6p+10}\right\} = L^{-1}\left\{\frac{1}{p^2-6p+9+1}\right\} = L^{-1}\left\{\frac{1}{(p-3)^2+1}\right\}$$

$$= e^{3t}\, L^{-1}\left\{\frac{1}{p^2+1}\right\} = e^{3t}\sin t$$

$$\boxed{L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t}$$

$$\therefore L^{-1}\left\{\frac{1}{p^2-6p+10}\right\} = e^{3t}\sin t$$

6. Find $L^{-1}\left\{\frac{1}{p^2-6p+9}\right\} = L^{-1}\left\{\frac{1}{(p-3)^2}\right\} = e^{3t}\, L^{-1}\left\{\frac{1}{p^2}\right\}$

$$= e^{3t}\cdot t$$

7. Find $L^{-1}\left\{\frac{1}{p^2+8p+16}\right\} = L^{-1}\left\{\frac{1}{(p+4)^2}\right\} = e^{-4t}\, L^{-1}\left\{\frac{1}{p^2}\right\}$

$$= e^{-4t}\cdot t$$

8. Evaluate $L^{-1}\left\{\frac{3p-2}{p^2-4p+20}\right\}$

Sol:

$$L^{-1}\left\{\frac{3p-2}{p^2-4p+20}\right\} = L^{-1}\left\{\frac{3(p-2)+4}{(p-2)^2+16}\right\}$$

$$= e^{2t}\, L^{-1}\left\{\frac{3p+4}{p^2+16}\right\}$$

$$= e^{2t}\, L^{-1}\left\{\frac{3p}{p^2+16} + \frac{4}{p^2+16}\right\}$$

$$= e^{2t}\left\{3L^{-1}\left\{\frac{p}{p^2+4^2}\right\} + 4\,L^{-1}\left\{\frac{1}{p^2+4^2}\right\}\right\}$$

$$= e^{2t}\left\{3\cos 4t + \sin 4t\right\}$$

9. Find $L^{-1}\left\{\frac{p}{(p+1)^5}\right\} = L^{-1}\left\{\frac{p+1-1}{(p+1)^5}\right\}$

$$= L^{-1}\left\{\frac{p+1}{(p+1)^5} - \frac{1}{(p+1)^5}\right\} = L^{-1}\left\{\frac{1}{(p+1)^4}\right\} - L^{-1}\left\{\frac{1}{(p+1)^5}\right\}$$

$$= e^{-t}\, L^{-1}\left\{\frac{1}{p^4}\right\} - e^{-t}\, L^{-1}\left\{\frac{1}{p^5}\right\}$$

$$= e^{-t}\left[\frac{t^3}{3!} - \frac{t^4}{4!}\right] = \frac{e^{-t}}{24}\left[4t^3 - t^4\right]$$

10. Find $L^{-1}\left\{\frac{p}{(p+1)^{5/2}}\right\}$

$$L^{-1}\left\{\frac{p}{(p+1)^{5/2}}\right\} = L^{-1}\left\{\frac{(p+1)-1}{(p+1)^{5/2}}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{p-1}{p^{5/2}}\right\} = e^{-t} L^{-1}\left\{\frac{p}{p^{5/2}} - \frac{1}{p^{5/2}}\right\}$$

$$= e^{-t}\left\{L^{-1}\left(\frac{1}{p^{3/2}}\right) - L^{-1}\left(\frac{1}{p^{5/2}}\right)\right\}$$

$$= e^{-t}\left\{\frac{t^{1/2}}{\Gamma(\frac{1}{2}+1)} - \frac{t^{3/2}}{\Gamma(\frac{3}{2}+1)}\right\}$$

$$= e^{-t}\left\{\frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} - \frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\right\}$$

$$= e^{-t}\left\{\frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{4t^{3/2}}{3\sqrt{\pi}}\right\}$$

$$= \frac{2\sqrt{t}}{\sqrt{\pi}} e^{-t}\left\{1 - \frac{2t}{3}\right\}_{11}$$

11. Find $L^{-1}\left\{\frac{3p+2}{4p^2+12p+9}\right\}$

$$L^{-1}\left\{\frac{(3p+2)}{4p^2+12p+9}\right\} = L^{-1}\left\{\frac{3p+\frac{9}{2}-\frac{5}{2}}{4p^2+4\cdot 2\cdot\frac{3p}{2}+4\cdot\frac{9}{4}}\right\}$$

$$= L^{-1}\left\{\frac{3(p+\frac{3}{2})-5/2}{4(p+\frac{3}{2})^2}\right\}$$

$$= e^{-3/2\,t} L^{-1}\left\{\frac{3p-5/2}{4p^2}\right\}$$

$$= e^{-3/2\,t} L^{-1}\left\{\frac{3}{4p} - \frac{5/2}{4p^2}\right\}$$

$$= e^{-\frac{3}{2}t}\left\{\frac{3}{4} L^{-1}\left(\frac{1}{p}\right) - \frac{5}{8} L^{-1}\left(\frac{1}{p^2}\right)\right\}$$

$$= e^{-\frac{3}{2}t}\left\{\frac{3}{4} - \frac{5}{8}t\right\}$$

$$\therefore L^{-1}\left\{\frac{3p+2}{4p^2+12p+9}\right\} = e^{-\frac{3}{2}t}\left\{\frac{3}{4} - \frac{5}{8}t\right\}$$

12. Find $L^{-1}\left\{\frac{1}{\sqrt{2p+3}}\right\}$

Sol:

$$L^{-1}\left\{\frac{1}{\sqrt{2p+3}}\right\} = L^{-1}\left\{\frac{1}{(2p+3)^{\frac{1}{2}}}\right\} = L^{-1}\left\{\frac{1}{2^{\frac{1}{2}}\left[p+\frac{3}{2}\right]^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\sqrt{2}} L^{-1}\left\{\frac{1}{(p+\frac{3}{2})^{\frac{1}{2}}}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} L^{-1}\left\{\frac{1}{p^{1/2}}\right\}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{3}{2}t} \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} = \frac{1}{\sqrt{2}} e^{-3/2t} \cdot \frac{1}{\sqrt{\pi t}}$$

13. Find $L^{-1}\left\{\frac{p+1}{p^2+6p+25}\right\}$

$$L^{-1}\left\{\frac{p+1}{p^2+6p+25}\right\} = L^{-1}\left\{\frac{p+3-2}{(p+3)^2+16}\right\}$$

$$= e^{-3t} L^{-1}\left\{\frac{p-2}{p^2+16}\right\}$$

$$= e^{-3t}\left\{L^{-1}\left(\frac{p}{p^2+16}\right) - 2L^{-1}\left(\frac{1}{p^2+16}\right)\right\}$$

$$= e^{-3t}\left\{\cos 4t - \frac{2}{4}\sin 4t\right\}$$

$$= e^{-3t}\left\{\cos 4t - \frac{1}{2}\sin 4t\right\}$$

14. Evaluate $L^{-1}\left\{\frac{p+b}{(p+b)^2+a^2}\right\}$

$$L^{-1}\left\{\frac{p+b}{(p+b)^2+a^2}\right\} = e^{-bt} L^{-1}\left\{\frac{p}{p^2+a^2}\right\}$$

$$= e^{-bt}\cos at$$

= Ans.

15. Evaluate $L^{-1}\left\{\frac{3p+7}{p^2-2p-3}\right\}$

consider

$$\frac{3p+7}{p^2-2p-3} = \frac{3p+7}{(p-3)(p+1)} = \frac{A}{p-3} + \frac{B}{p+1} \longrightarrow (1)$$

$$\Rightarrow \quad 3p+7 = A(p+1) + B(p-3)$$

put $p=-1$	put $p=3$
$-3+7 = B(-1-3)$	$3\times 3+7 = A(3+1)$
$4 = -4B$	$16 = 4A$
$B = -1$	$A = 4$

using the values of A, B in eq (1), we get

$$\frac{3p+7}{p^2-2p-3} = \frac{4}{p-3} - \frac{1}{p+1}$$

Now $L^{-1}\left\{\frac{3p+7}{p^2-2p-3}\right\} = L^{-1}\left\{\frac{4}{p-3} - \frac{1}{p+1}\right\}$

$$= 4e^{3t} - e^{-t} \quad = \text{Ans.}$$

16. Evaluate $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$

consider

$$\frac{1}{(p+1)(p-2)} = \frac{A}{p+1} + \frac{B}{p-2} \longrightarrow (1)$$

$$\Rightarrow \quad 1 = A(p-2) + B(p+1)$$

put $p=2$	put $p=-1$
$1 = 3B$	$1 = A(-1-2)$
$B = \frac{1}{3}$	$A = -\frac{1}{3}$

using the values of A, B in eq (1), we get

$$\frac{1}{(p+1)(p-2)} = \frac{-1/3}{p+1} + \frac{\frac{1}{3}}{p-2}$$

Now

$$L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\} = L^{-1}\left\{\frac{-\frac{1}{3}}{p+1} + \frac{\frac{1}{3}}{p-2}\right\}$$

$$= -\frac{1}{3}L^{-1}\left\{\frac{1}{p+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{p-2}\right\}$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}$$

17. Evaluate $L^{-1}\left\{\dfrac{p-1}{(p+3)(p^2+2p+2)}\right\}$

Sol:

$$\frac{p-1}{(p+3)(p^2+2p+2)} = \frac{A}{(p+3)} + \frac{Bp+C}{p^2+2p+2} \longrightarrow ①$$

$$\Rightarrow p-1 = A(p^2+2p+2) + (Bp+C)(p+3)$$

$$= Ap^2 + 2Ap + 2A + Bp^2 + 3Bp + Cp + 3C$$

$$= (A+B)p^2 + (2A+3B+C)p + (2A+3C)$$

Equating the coeff of p^2, p and constant terms

$A+B=0$

$\Rightarrow A=-B$

$2A+3B+C=1$

$-2B+3B+C=1$

$B+C=1$

$C=1-B$

$2A+3C=-1$

$-2B+3(1-B)=-1$

$-2B+3-3B=-1$

$\Rightarrow -5B=-4$

$\Rightarrow B=\frac{4}{5}$

$\therefore C = 1-\frac{4}{5} = \frac{1}{5}$

$\therefore A = -\frac{4}{5}$

using the values of A, B, C in eq ①, we get

$$\frac{p-1}{(p+3)(p^2+2p+2)} = \frac{-4/5}{p+3} + \frac{(4/5)p+(1/5)}{p^2+2p+2}$$

Now

$$L^{-1}\left\{\frac{p-1}{(p+3)(p^2+2p+2)}\right\} = -\frac{4}{5}L^{-1}\left\{\frac{1}{p+3}\right\} + \frac{1}{5}L^{-1}\left\{\frac{4p+1}{p^2+2p+2}\right\}$$

$$= -\frac{4}{5}e^{-3t} + \frac{1}{5}L^{-1}\left\{\frac{4(p+1)-3}{(p+1)^2+1}\right\}$$

$$= -\frac{4}{5}e^{-3t} + \frac{1}{5}e^{-t}L^{-1}\left\{\frac{4p-3}{p^2+1}\right\}$$

$$= -\frac{4}{5}e^{-3t} + \frac{1}{5}e^{-t}\left\{4L^{-1}\left\{\frac{p}{p^2+1}\right\} - 3L^{-1}\left\{\frac{1}{p^2+1}\right\}\right\}$$

$$= -\frac{4}{5}e^{-3t} + \frac{1}{5}e^{-t}\{4\cos t - 3\sin t\}$$ = Ans.

18. Evaluate $L^{-1}\left\{\dfrac{p}{(p^2-2p+2)(p^2+2p+2)}\right\}$

Sol: Consider

$$\frac{p}{(p^2-2p+2)(p^2+2p+2)} = \frac{Ap+B}{p^2-2p+2} + \frac{Cp+D}{(p^2+2p+2)} \rightarrow ①$$

$$\Rightarrow p = (Ap+B)(p^2+2p+2) + (Cp+D)(p^2-2p+2)$$

$\Rightarrow p = Ap^3 + 2Ap^2 + 2Ap + Bp^2 + 2Bp + 2B + Cp^3 - 2Cp^2$
$+ 2Cp + Dp^2 - 2Dp + 2D$

$= (A+C)p^3 + (2A+B-2C+D)p^2 + (2A+2B+2C-2D)p + (2B+2D)$

Equating the coefficients of p^3, p^2, p and constant terms, we get

$A + C = 0 \rightarrow ②$ $\quad 2A + B - 2C + D = 0 \rightarrow ③$ $\quad 2A + 2B + 2C - 2D = 1 \rightarrow ④$

$\Rightarrow 2B + 2D = 0 \rightarrow ⑤$

using ②, ④ $\Rightarrow 2B - 2D = 1 \rightarrow ⑥$

Add ⑤, ⑥ $\Rightarrow 4B = 1 \Rightarrow B = \frac{1}{4} \Rightarrow D = -\frac{1}{4}$

eq ③ $\Rightarrow 2A - 2C = 0 \Rightarrow A - C = 0 \rightarrow ⑦$

Add ②, ⑦ $\Rightarrow 2A = 0 \Rightarrow A = 0 \Rightarrow C = 0$

$\therefore A = 0,\ B = \frac{1}{4},\ C = 0,\ D = -\frac{1}{4}$

using the values of A, B, C, D in eq ①, we get

$$\frac{p}{(p^2-2p+2)(p^2+2p+2)} = \frac{1}{4(p^2-2p+2)} - \frac{1}{4(p^2+2p+2)}$$

$$L^{-1}\left\{\frac{p}{(p^2-2p+2)(p^2+2p+2)}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{p^2-2p+2}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{p^2+2p+2}\right\}$$

$$= \frac{1}{4}L^{-1}\left\{\frac{1}{(p-1)^2+1}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\}$$

$$= \frac{e^t}{4}L^{-1}\left\{\frac{1}{p^2+1}\right\} - \frac{e^{-t}}{4}L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= \frac{e^t}{4}\sin t - \frac{e^{-t}}{4}\sin t$$

$$= \frac{1}{2}\sin t\left[\frac{e^t - e^{-t}}{2}\right]$$

$$= \frac{1}{2}\sin t \sinh t$$

$$\therefore L^{-1}\left\{\frac{p}{(p^2-2p+2)(p^2+2p+2)}\right\} = \frac{1}{2}\sin t \sinh t$$

19 Evaluate $L^{-1}\left\{\frac{3p+1}{(p+1)^4}\right\}$

Consider

$$\frac{3p+1}{(p+1)^4} = \frac{A}{p+1} + \frac{B}{(p+1)^2} + \frac{C}{(p+1)^3} + \frac{D}{(p+1)^4} \rightarrow ①$$

$$3P+1 = A(P+1)^3 + B(P+1)^2 + C(P+1) + D$$
$$= A(P^3+3P^2+3P+1) + B(P^2+2P+1) + CP + C + D$$
$$= AP^3 + (3A+B)P^2 + (3A+2B+C)P + (A+B+C+D)$$

Equating the coeff of P^3, P^2, P and constant terms

$A = 0$ | $3A+B=0 \Rightarrow B=0$ | $3A+2B+C=\cancel{0}3 \Rightarrow C=3$ | $A+B+C+D=1$, $D=1-3$, $D=-2$

using the values of A, B, C, D in eq ①, we get

$$\frac{3P+1}{(P+1)^4} = \frac{3}{(P+1)^3} - \frac{2}{(P+1)^4}$$

Then $L^{-1}\left\{\frac{3P+1}{(P+1)^4}\right\} = 3L^{-1}\left\{\frac{1}{(P+1)^3}\right\} - 2L^{-1}\left\{\frac{1}{(P+1)^4}\right\}$

$$= 3e^{-t}L^{-1}\left\{\frac{1}{P^3}\right\} - 2e^{-t}L^{-1}\left\{\frac{1}{P^4}\right\}$$
$$= 3e^{-t}\frac{t^2}{2} - 2e^{-t}\frac{t^3}{3!}$$
$$= t^2e^{-t}\left[\frac{9-2t}{6}\right]$$

$$\therefore L^{-1}\left\{\frac{3P+1}{(P+1)^4}\right\} = t^2e^{-t}\left[\frac{9-2t}{6}\right] \text{ — Ans.}$$

20. Evaluate $L^{-1}\left\{\frac{6P}{P^2+2P-3}\right\}$

consider

$$\frac{6P}{P^2+2P-3} = \frac{6P}{(P+3)(P-1)} = \frac{A}{P+3} + \frac{B}{P-1} \longrightarrow ①$$

$$\Rightarrow 6P = A(P-1) + B(P+3)$$

put $P=1$: $6 = 4B \Rightarrow B = \frac{3}{2}$ | put $P=-3$: $-18 = -4A$, $A = \frac{9}{2}$

using A, B in eq ①, we get

$$\frac{6P}{P^2+2P-3} = \frac{3}{2(P+3)} + \frac{9}{2(P-1)} \Rightarrow L^{-1}\left\{\frac{6P}{P^2+2P-3}\right\} = L^{-1}\left\{\frac{3}{2(P+3)} + \frac{9}{2(P-1)}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{6P}{P^2+2P-3}\right\} = \frac{3}{2}e^{-3t} + \frac{9}{2}e^{t} \text{ — Ans}$$

21. Evaluate $L^{-1}\left\{\frac{p+b}{(p+b)^2+a^2}\right\}$

$$L^{-1}\left\{\frac{p+b}{(p+b)^2+a^2}\right\} = e^{-bt} L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = e^{-bt}\cos at$$

22. Evaluate $L^{-1}\left\{\frac{2p-1}{p^3-p}\right\}$

consider

$$\frac{2p-1}{p^3-p} = \frac{2p-1}{p(p^2-1)} = \frac{2p-1}{p(p-1)(p+1)}$$

$$= \frac{A}{p} + \frac{B}{p-1} + \frac{C}{p+1} \longrightarrow (1)$$

$$\Rightarrow 2p-1 = A(p^2-1) + Bp(p+1) + Cp(p-1)$$

put $p=0$	put $p=1$	put $p=-1$
$-1=-A$	$2B=1$	$-3=2C$
$\Rightarrow A=1$	$B=\frac{1}{2}$	$C=\frac{-3}{2}$

using the values of A, B, C in eq (1), we get

$$\frac{2p-1}{p^3-p} = \frac{1}{p} + \frac{1}{2(p-1)} + \frac{-3}{2(p+1)}$$

Now $L^{-1}\left\{\frac{2p-1}{p^3-p}\right\} = L^{-1}\left\{\frac{1}{p}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p-1}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{p+1}\right\}$

$$= 1 + \frac{1}{2}e^{t} - \frac{3}{2}e^{-t}$$

23. Evaluate $L^{-1}\left\{\frac{11p^2-2p+5}{(p+1)(p-2)(2p-1)}\right\}$

Consider

$$\frac{11p^2-2p+5}{(p+1)(p-2)(2p-1)} = \frac{A}{p+1} + \frac{B}{p-2} + \frac{C}{2p-1} \longrightarrow (1)$$

$$\Rightarrow 11p^2-2p+5 = A(p-2)(2p-1) + B(p+1)(2p-1) + C(p+1)(p-2)$$

put $p=2 \Rightarrow 11\times4-2\times2+5 = B(2+1)(4-1)$

$\Rightarrow 45 = 9B$

$\Rightarrow B=5$

put $p=-1 \Rightarrow 11+2+5 = A(-1-2)(-2-1)$

$\Rightarrow 18 = 9A$

$\Rightarrow A=2$

put $2P-1=0 \Rightarrow P=\frac{1}{2}$

Then $\frac{11}{4}-1+5 = C(\frac{1}{2}+1)(\frac{1}{2}-2)$

$\Rightarrow \frac{27}{4} = C(\frac{3}{2})(-\frac{3}{2}) \Rightarrow C = -\frac{27}{9} \Rightarrow C=-3$

using the values of A, B, C in eq ①, we get

$$\frac{11P^2-2P+5}{(P+1)(P-2)(2P-1)} = \frac{2}{P+1} + \frac{5}{P-2} - \frac{3}{(2P-1)}$$

$$L^{-1}\left\{\frac{11P^2-2P+5}{(P+1)(P-2)(2P-1)}\right\} = 2L^{-1}\left\{\frac{1}{P+1}\right\} + 5L^{-1}\left\{\frac{1}{P-2}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{P-\frac{1}{2}}\right\}$$

$$= 2e^{-t} + 5e^{2t} - \frac{3}{2}e^{\frac{1}{2}t}$$

24. Evaluate $L^{-1}\left\{\frac{P+5}{(P+2)(P^2+4)}\right\}$

consider

$$\frac{P+5}{(P+2)(P^2+4)} = \frac{A}{(P+2)} + \frac{BP+C}{P^2+4} \longrightarrow ①$$

$\Rightarrow P+5 = A(P^2+4) + (BP+C)(P+2)$

$\Rightarrow P+5 = AP^2+4A+BP^2+2BP+CP+2C$

$= (A+B)P^2 + (2B+C)P + (4A+2C)$

Equating the coeff of P^2, P and constant terms

$A+B=0$

$A=-B$

$2B+C=1$

$4A+2C=5$

$-4B+2C=5$

$4B+2C=2$

$4C=7 \Rightarrow C=\frac{7}{4}$

Then $2B = 1-\frac{7}{4} = -\frac{3}{4}$

$B = -\frac{3}{8} \Rightarrow A=\frac{3}{8}$

using the values of A, B, C in eq ①

$$\frac{P+5}{(P+2)(P^2+4)} = \frac{3}{8(P+2)} + \frac{-\frac{3}{8}P+\frac{7}{4}}{P^2+4}$$

Now

$$L^{-1}\left\{\frac{P+5}{(P+2)(P^2+4)}\right\} = \frac{3}{8}L^{-1}\left\{\frac{1}{P+2}\right\} - \frac{3}{8}L^{-1}\left\{\frac{P}{P^2+4}\right\} + \frac{7}{4}L^{-1}\left\{\frac{1}{P^2+4}\right\}$$

$$= \frac{3}{8}e^{-2t} - \frac{3}{8}\cos 2t + \frac{7}{8}\sin 2t$$ = Ans.

Evaluate the following

$L^{-1}\left\{\dfrac{p+1}{p^2+6p+25}\right\}$

$L^{-1}\left\{\dfrac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\}$

$L^{-1}\left\{\dfrac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}\right\}$

$L^{-1}\left\{\dfrac{4p+5}{(p-1)^2(p+2)}\right\}$

$L^{-1}\left\{\dfrac{1}{(p+2)(p-1)^2}\right\}$

$L^{-1}\left\{\dfrac{3p^3-3p^2-40p+36}{(p^2-4)^2}\right\}$

$L^{-1}\left\{\dfrac{p^2-2p+3}{(p-1)^2(p+1)}\right\}$

$L^{-1}\left\{\dfrac{p+2}{p^2-2p+5}\right\}$

$L^{-1}\left\{\dfrac{p^2}{(p+2)^2}\right\}$

$L^{-1}\left\{\dfrac{2p^3+2p^2+4p+1}{(p^2+1)(p^2+p+1)}\right\}$

Problems Based on unit Step Function (OR) Heaviside's Step Function:

1. Find $L^{-1}\left\{\dfrac{e^{-5p}}{(p-2)^4}\right\}$

Now $L^{-1}\left\{\dfrac{1}{(p-2)^4}\right\} = e^{2t}\, L^{-1}\left\{\dfrac{1}{p^4}\right\} = e^{2t}\dfrac{t^3}{3!}$

$\therefore L^{-1}\left\{\dfrac{1}{(p-2)^4}\right\} = \dfrac{e^{2t}}{6}t^3$

Now $L^{-1}\left\{\dfrac{e^{-5p}}{(p-2)^4}\right\} = \begin{cases} \frac{1}{6}e^{2(t-5)}(t-5)^3 & \text{if } t>5 \\ 0 & \text{if } t<5 \end{cases}$

$= \frac{1}{6}e^{2(t-5)}(t-5)^3 H(t-5)$, in terms of Heaviside unit step function

2. Find $L^{-1}\left\{\frac{e^{-4p}}{(p-3)^4}\right\}$

$$L^{-1}\left\{\frac{1}{(p-3)^4}\right\} = e^{3t} L^{-1}\left\{\frac{1}{p^4}\right\} = e^{3t}\frac{t^3}{3!}$$

Now

$$L^{-1}\left\{\frac{e^{-4p}}{(p-3)^4}\right\} = \begin{cases} \frac{1}{6} e^{3(t-4)} (t-4)^3 & \text{if } t>4 \\ 0 & \text{if } t<4 \end{cases}$$

$$= \frac{1}{6}(t-4)^3 e^{3(t-4)} H(t-4),$$ in terms of Heaviside unit step function

3. Find $L^{-1}\left\{\frac{e^{-3p}}{p^3}\right\}$

$$L^{-1}\left\{\frac{1}{p^3}\right\} = \frac{t^2}{2!}$$

$$L^{-1}\left\{\frac{e^{-3p}}{p^3}\right\} = \begin{cases} \frac{1}{2}(t-3)^2 & \text{if } t>3 \\ 0 & \text{if } t<3 \end{cases}$$

$$= \frac{1}{2}(t-3)^2 H(t-3),$$ in terms of Heaviside unit step function.

4. Find $L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\}$

$$L^{-1}\left\{\frac{1}{(p+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{1}{p^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{1}{p^{\frac{3}{2}+1}}\right\}$$

$$= e^{-4t}\frac{t^{3/2}}{\Gamma(\frac{3}{2}+1)} = e^{-4t}\frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{4}{3\sqrt{\pi}} e^{-4t} t^{3/2}$$

Now

$$L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\} = e^4 L^{-1}\left\{\frac{e^{-3p}}{(p+4)^{5/2}}\right\}$$

$$= \frac{4}{3\sqrt{\pi}}\begin{cases} e^{-4(t-3)}\cdot(t-3)^{\frac{3}{2}} & \text{if } t\geq 3 \\ 0 & \text{if } t<3 \end{cases}$$

$$= \frac{4}{3\sqrt{\pi}} e^{-4(t-3)} (t-3)^{\frac{3}{2}} H(t-3)$$

= Ans.

5. Find $F(t)$, $F(t) = L^{-1}\left\{\frac{3}{p} - \frac{4e^{-p}}{p^2} + \frac{4e^{-3p}}{p^2}\right\}$

$$F(t) = L^{-1}\left\{\frac{3}{p} - \frac{4e^{-p}}{p^2} + \frac{4e^{-3p}}{p^2}\right\}$$

$$= 3L^{-1}\left\{\frac{1}{p}\right\} - 4L^{-1}\left\{\frac{e^{-p}}{p^2}\right\} + 4L^{-1}\left\{\frac{e^{-3p}}{p^2}\right\}$$

$$= 3 - 4(t-1) + 4(t-3) \quad (\text{OR})$$

$$= 3 - 4(t-1)H(t-1) + 4(t-3)H(t-3),$$

in terms of Heaviside unit step function

6. Find $L^{-1}\left\{\frac{e^{-p\pi}}{p^2+1}\right\}$

Now $L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t$

Now $L^{-1}\left\{\frac{e^{-p\pi}}{p^2+1}\right\} = \begin{cases} \sin(t-\pi) & \text{if } t \geq \pi \\ 0 & \text{if } t < \pi \end{cases}$

$$= \sin(t-\pi)\, H(t-\pi)$$

$$= -\sin(\pi - t)\, H(t-\pi),$$ in terms of Heaviside unit step function

7. Find $L^{-1}\left\{\frac{(p+1)e^{-\pi p}}{p^2+p+1}\right\}$

Now

$$L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\} = L^{-1}\left\{\frac{(p+\frac{1}{2}) + \frac{1}{2}}{p^2 + 2\cdot\frac{1}{2}\cdot p + (\frac{1}{2})^2 + \frac{3}{4}}\right\}$$

$$= L^{-1}\left\{\frac{(p+\frac{1}{2}) + \frac{1}{2}}{(p+\frac{1}{2})^2 + \frac{3}{4}}\right\}$$

$$= e^{-\frac{1}{2}t}\, L^{-1}\left\{\frac{p+\frac{1}{2}}{p^2+\frac{3}{4}}\right\}$$

$$= e^{-\frac{1}{2}t}\left[L^{-1}\left\{\frac{p}{p^2+(\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p^2+(\frac{\sqrt{3}}{2})^2}\right\}\right]$$

$$= e^{-\frac{1}{2}t}\left\{\cos(\tfrac{\sqrt{3}}{2})t + \frac{1}{2}\times\frac{2}{\sqrt{3}}\sin(\tfrac{\sqrt{3}}{2})t\right\}$$

Now.

$$L^{-1}\left\{\frac{(p+1)e^{-p\pi}}{p^2+p+1}\right\} = e^{-\frac{1}{2}(t-\pi)}\left\{\cos(\tfrac{\sqrt{3}}{2})(t-\pi) + \frac{1}{\sqrt{3}}\sin(\tfrac{\sqrt{3}}{2})(t-\pi)\right.$$

$$= e^{-\frac{1}{2}(t-\pi)}\left[\cos\tfrac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}}\sin(\tfrac{\sqrt{3}}{2})(t-\pi)\right]\cdot H(t-\pi)$$

in terms of H.S.U.S.F.

8. Prove that $L^{-1}\left\{\frac{p e^{-\frac{2p\pi}{3}}}{p^2+9}\right\} = \cos 3(t-\frac{2\pi}{3})\, H(t-\frac{2\pi}{3})$

$$L^{-1}\left\{\frac{p}{p^2+9}\right\} = L^{-1}\left\{\frac{p}{p^2+3^2}\right\} = \cos 3t$$

Now

$$L^{-1}\left\{\frac{p e^{-\frac{2p\pi}{3}}}{p^2+9}\right\} = \begin{cases}\cos 3(t-\frac{2\pi}{3}) & \text{if } t \geq \frac{2\pi}{3}\\ 0 & \text{if } t < \frac{2\pi}{3}\end{cases}$$

$$= \cos 3(t-\tfrac{2\pi}{3})\, H(t-\tfrac{2\pi}{3})$$

in terms of Heaviside unit step function.

9. Prove that $L^{-1}\left\{\frac{p e^{-ap}}{p^2-b^2}\right\} = \cosh b(t-a)\, H(t-a)$

10. Find $L^{-1}\left\{\frac{3(1+e^{-p\pi})}{p^2+9}\right\}$

Inverse Laplace Transform of Derivatives:

If $L^{-1}\{f(P)\} = F(t)$, then $L^{-1}\{f^{n}(P)\} = L^{-1}\left\{\frac{d^n}{dp^n} f(P)\right\} = (-1)^n t^n F(t)$

Proof: we know that

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(P) = (-1)^n \frac{d^n}{dp^n} L\{F(t)\}$$

$$\Rightarrow t^n F(t) = L^{-1}\left\{(-1)^n \frac{d^n}{dp^n} f(P)\right\} = (-1)^n L^{-1}\left\{\frac{d^n}{dp^n} f(P)\right\}$$

$$\therefore (-1)^n t^n F(t) = L^{-1}\left\{\frac{d^n}{dp^n} f(P)\right\}$$

Inverse Laplace Transform of Integral:

we know that

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)\,dx, \text{ Provided } \lim_{t\to 0} \frac{F(t)}{t} \text{ exists}$$

$$\boxed{\therefore \frac{F(t)}{t} = L^{-1}\left\{\int_p^\infty f(x)\,dx\right\}}$$

Multiplication by powers of P:

If $L^{-1}\{f(P)\} = F(t)$ and $F(0)=0$, then $L^{-1}\{p f(P)\} = \{F'(t)\}$

Proof: we know that

$$L\{F'(t)\} = p f(P) - F(0) \quad \& \quad F(0)=0$$

$$\Rightarrow L\{F'(t)\} = p f(P)$$

$$\Rightarrow \boxed{F'(t) = L^{-1}\{p f(P)\}}$$

Division by powers of P:

If $F(t)$ is sectionally continuous and of exponential order "a" such that $\lim_{t\to 0} \frac{F(t)}{t}$ exists,

Then $L^{-1}\left\{\frac{f(P)}{P}\right\} = \int_0^t F(t)\,dt$, where $p>0$.

Proof:

Let $G(t) = \int_0^t F(t)\,dt = \int_0^t F(x)\,dx$

Then $G'(t) = F(t)$ and $G(0)=0$

$$\because L\{G'(t)\} = p\,g(P) - G(0)$$

$$\Rightarrow L\{F(t)\} = p\,g(P)$$

$$\Rightarrow f(P) = p\,g(P) \quad , \quad L\{G(t)\} = g(P)$$

$$\Rightarrow \frac{f(P)}{P} = L\{G(t)\}$$

$$\Rightarrow L^{-1}\left\{\frac{f(P)}{P}\right\} = G(t)$$

$$\Rightarrow L^{-1}\left\{\frac{f(P)}{P}\right\} = \int_0^t F(t)\,dt$$

Hence proved.

Note: Let $G(t) = \int_0^t \int_0^y F(x)\, dx\, dy$

$$\Rightarrow G'(t) = \int_0^t F(x)\, dx \Rightarrow G''(t) = F(t) \text{ and } G(0) = G'(0) = 0$$

$$\because L\{G''(t)\} = p^2 g(p) - pG(0) - G'(0)$$

$$\Rightarrow L\{F(t)\} = p^2 g(p)$$

$$\Rightarrow f(p) = p^2 L\{G(t)\}$$

$$\Rightarrow \frac{f(p)}{p^2} = L\{G(t)\}$$

$$\Rightarrow L^{-1}\left\{\frac{f(p)}{p^2}\right\} = G(t) = \int_0^t \int_0^y F(x)\, dx\, dy$$

lly $L^{-1}\left\{\frac{f(p)}{p^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t F(t)\, dt\, dt \dots dt$ (n times)

Convolution:

Let $F(t)$ and $G(t)$ be two functions of class A. Then the convolution of $F(t)$ and $G(t)$ is denoted by $F*G$ and defined as

$$\boxed{F*G = \int_0^t F(x)\, G(t-x)\, dx}$$

Note: This relation $F*G$ is also called the resultant (or) falting of F and G.

Properties of convolution:

1) $F*G$ is commutative i.e $F*G = G*F$

$$\int_0^t F(x)\, G(t-x)\, dx = \int_0^t G(x)\, F(t-x)\, dx$$

2) $F*G$ is associative

i.e $(F*G)*H = F*(G*H)$

3) $F*G$ is distributive w.r.to addition

i.e $F*(G+H) = F*G + F*H$

Convolution Theorem:

Let $F(t)$ and $G(t)$ be any two functions of class A and $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$,

then $L^{-1}\{f(p)\, g(p)\} = \int_0^t F(u)\, G(t-u)\, du = F*G$

Proof:

Let $F(t)$ and $G(t)$ be two functions of class A.

$$L\{F(t)\} = f(P) \text{ and } L\{G(t)\} = g(P)$$

i.e $F(t) = L^{-1}\{f(P)\}$ and $G(t) = L^{-1}\{g(P)\}$

we have to prove that

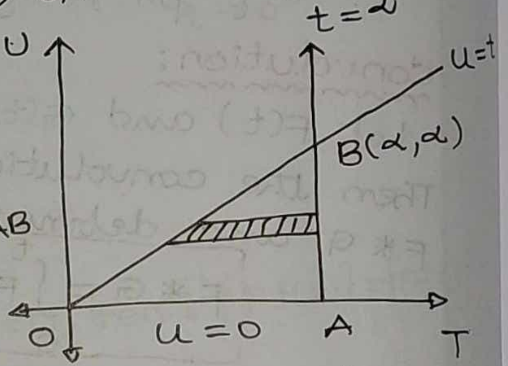
$$L^{-1}\{f(P)g(P)\} = \int_0^t F(u)\, G(t-u)\, du = F * G$$

To prove this, define $H(t) = \int_0^t F(u)\, G(t-u)\, du$

Then $L\{H(t)\} = \int_0^\infty e^{-pt} H(t)\, dt$

$$= \int_0^\infty e^{-pt} \left[\int_0^t F(u)\, G(t-u)\, du \right] dt$$

$$= \int_0^\infty \int_0^t e^{-pt} F(u)\, G(t-u)\, du\, dt \rightarrow ①$$



To change the order of limits:

limits : $t=0$, $t=\infty$

$u=0$, $u=t$

Range of integration is ΔOAB

Draw an elementary strip parallel to T-axis, one end of the strip lies on $t=u$ and the other end lies on $t=\alpha$.

For this strip u varies from $u=0$ to $u=\alpha$

Then eq① ⇒

$$L\{H(t)\} = \int_{t=0}^{\infty} \int_{u=0}^{\infty} e^{-pt} F(u)\, G(t-u)\, du\, dt$$

$$= \int_0^\infty F(u)\, du \int_0^\infty e^{-pt} G(t-u)\, dt$$

put $t = u+y$

$dt = dy$

$$= \int_0^\infty F(u)\, du \int_0^\infty e^{-p(u+y)} G(y)\, dy$$

$$= \int_0^\infty F(u)\, du \int_0^\infty e^{-pu} e^{-py} G(y)\, dy = \int_0^\infty F(u)\, e^{-pu}\, du \cdot \int_0^\infty e^{-py} G(y)\, dy$$

$$= L\{F(u)\}\, L\{G(y)\}$$

$$= f(P)\, g(P)$$

$\therefore L\{H(t)\} = f(p)g(p)$

$\Rightarrow H(t) = L^{-1}\{f(P)g(P)\}$

$\Rightarrow \int_0^t F(u)\, G(t-u)\, du = L^{-1}\{f(P)g(P)\}$

Hence proved

Problems Based on Convolution Theorem:

1. Find $L^{-1}\left\{\frac{1}{(P+a)(P+b)}\right\}$

Let $f(P) = \frac{1}{P+a}$, $g(P) = \frac{1}{P+b}$

Then $L^{-1}\{f(P)\} = L^{-1}\left\{\frac{1}{P+a}\right\} = e^{-at} = F(t)$

$L^{-1}\{g(P)\} = L^{-1}\left\{\frac{1}{P+b}\right\} = e^{-bt} = G(t)$

By convolution Theorem

$$L^{-1}\{f(P)\,g(P)\} = \int_0^t F(x)\,G(t-x)\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(P+a)(P+b)}\right\} = \int_0^t e^{-ax} e^{-b(t-x)}\,dx$$

$$= \int_0^t e^{-ax} e^{-bt} e^{bx}\,dx$$

$$= e^{-bt}\int_0^t e^{(b-a)x}\,dx = e^{-bt}\left[\frac{e^{(b-a)x}}{b-a}\right]_{x=0}^{t}$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - e^{0}\right] = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{(P+a)(P+b)}\right\} = \frac{e^{-at} - e^{-bt}}{b-a}$$

2. Find $L^{-1}\left\{\frac{1}{(P-a)(P-b)}\right\}$

Sol: Let $f(P) = \frac{1}{P-a}$, $g(P) = \frac{1}{P-b}$

Then $L^{-1}\{f(P)\} = L^{-1}\left\{\frac{1}{P-a}\right\} = e^{at} = F(t)$

$L^{-1}\{g(P)\} = L^{-1}\left\{\frac{1}{P-b}\right\} = e^{bt} = G(t)$

By Convolution Theorem

$$L^{-1}\{f(P)\,g(P)\} = \int_0^t F(x)\,G(t-x)\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(P-a)(P-b)}\right\} = \int_0^t e^{ax} e^{b(t-x)}\,dx$$

$$= \int_0^t e^{ax} e^{bt} e^{-bx}\,dx$$

$$= e^{bt}\int_0^t e^{(a-b)x}\,dx = e^{bt}\left[\frac{e^{(a-b)x}}{a-b}\right]_{x=0}^{x=t}$$

$$= \frac{e^{bt}}{a-b}\left[e^{(a-b)t} - e^{0}\right] = \frac{e^{bt}}{a-b}\left[e^{at}e^{-bt} - 1\right]$$

$$= \frac{1}{a-b}\left[e^{at} - e^{bt}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{(P-a)(P-b)}\right\} = \frac{1}{a-b}\left[e^{at} - e^{bt}\right]$$

3. Evaluate the following by Convolution Theorem:

$L^{-1}\left\{\frac{1}{(P-1)(P-2)}\right\}$, $L^{-1}\left\{\frac{1}{(P+3)(P-4)}\right\}$, $L^{-1}\left\{\frac{1}{(P-3)(P+4)}\right\}$, $L^{-1}\left\{\frac{1}{(P-1)(P+2)}\right\}$

Note:

1. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
2. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
3. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
4. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
5. $\int \sin ax\, dx = -\frac{1}{a}\cos ax$
6. $\int \cos ax\, dx = \frac{1}{a}\sin ax$
7. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$
8. $\sin 2\theta = 2\sin\theta\cos\theta$
9. $\cos 0^\circ = 1$, $\sin 0^\circ = 0$
10. $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$
11. $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$
12. $\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[f'(x)\cdot\int g(x)\,dx\right]dx$

4. Find $L^{-1}\left\{\frac{P}{(P^2+a^2)^2}\right\}$ by using Convolution Theorem.

Sol: Let $f(P) = \frac{P}{P^2+a^2}$, $g(P) = \frac{1}{P^2+a^2}$

Then $L^{-1}\{f(P)\} = L^{-1}\left\{\frac{P}{P^2+a^2}\right\} = \cos at = F(t)$

$L^{-1}\{g(P)\} = L^{-1}\left\{\frac{1}{P^2+a^2}\right\} = \frac{1}{a}\sin at = G(t)$

By Convolution theorem, we have

$$L^{-1}\{f(P)g(P)\} = \int_0^t F(x)\,G(t-x)\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{P}{(P^2+a^2)^2}\right\} = \int_0^t \cos ax\cdot\frac{1}{a}\sin a(t-x)\,dx$$

$$= \frac{1}{a}\int_0^t \cos ax\,\sin[at-ax]\,dx$$

$$= \frac{1}{a}\int_0^t \cos ax\left[\sin at\cos ax - \cos at\sin ax\right]dx$$

$$= \frac{1}{a}\int_0^t [\sin at \cos^2 ax - \cos at \sin ax \cos ax]\, dx$$

$$= \frac{1}{a}\sin at \int_0^t \left(\frac{1+\cos 2ax}{2}\right) dx - \frac{1}{2a}\int_0^t \sin 2ax\, dx$$

$$= \frac{1}{a}\sin at \cdot \frac{1}{2}\left[x + \frac{1}{2a}\sin 2ax\right]_{x=0}^{t} + \frac{1}{4a^2}\cos at\,[\cos 2ax]_0^t$$

$$= \frac{1}{2a}\sin at\left[t + \frac{1}{2a}\sin 2at\right] + \frac{\cos at}{4a^2}[\cos 2at - 1]$$

$$= \frac{t}{2a}\sin at + \frac{1}{4a^2}[\sin 2at \sin at + \cos 2at \cos at] - \frac{\cos at}{4a^2}$$

$$= \frac{t}{2a}\sin at + \frac{1}{4a^2}\cos(2at - at) - \frac{1}{4a^2}\cos at$$

$$= \frac{t}{2a}\sin at + \frac{1}{4a^2}\cos at - \frac{1}{4a^2}\cos at$$

$$= \frac{t}{2a}\sin at$$

$$\therefore L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

5. Find $L^{-1}\left\{\frac{p^2}{(p^2+a^2)^2}\right\}$ by using convolution Theorem.

Sol: Let $f(p) = \frac{p}{p^2+a^2}$, $g(p) = \frac{p}{p^2+a^2}$

Then

$$L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at = F(t)$$

$$L^{-1}\{g(p)\} = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at = G(t)$$

By convolution Theorem, we have

$$L^{-1}\{f(p)g(p)\} = \int_0^t F(x)\, G(t-x)\, dx$$

$$\Rightarrow L^{-1}\left\{\frac{p^2}{(p^2+a^2)^2}\right\} = \int_0^t \cos ax \cos a(t-x)\, dx$$

$$= \int_0^t \cos ax \cos(at - ax)\, dx$$

$$= \int_0^t \cos ax\{\cos at \cos ax + \sin at \sin ax\}\, dx$$

$$= \int_0^t \cos^2 ax \cos at \, dx + \int_0^t \cos ax \sin at \sin at \, dx$$

$$= \cos at \int_0^t \left(\frac{1+\cos 2ax}{2}\right) dx + \frac{\sin at}{2} \int_0^t \sin 2ax \, dx$$

$$= \frac{\cos at}{2}\left[x + \frac{1}{2a}\sin 2ax\right]_{x=0}^{t} + \frac{\sin at}{-4a}\left[\cos 2ax\right]_{x=0}^{t}$$

$$= \frac{\cos at}{2}\left[t + \frac{1}{2a}\sin 2at\right] - \frac{1}{4a}\sin at\left[\cos 2at - 1\right]$$

$$= \frac{t}{2}\cos at + \frac{1}{4a}\sin 2at \cos at - \frac{1}{4a}\cos 2at \sin at + \frac{1}{4a}\sin at$$

$$= \frac{t}{2}\cos at + \frac{1}{4a}\left[\sin 2at \cos at - \cos 2at \sin at\right] + \frac{1}{4a}\sin at$$

$$= \frac{t}{2}\cos at + \frac{1}{4a}\sin at + \frac{1}{4a}\sin at$$

$$= \frac{t}{2}\cos at + \frac{1}{2a}\sin at$$

$$\therefore L^{-1}\left\{\frac{p^2}{(p^2+a^2)^2}\right\} = \frac{t}{2}\cos at + \frac{1}{2a}\sin at \quad = \text{Ans.}$$

6. Find $L^{-1}\left\{\frac{p^2}{(p^2+4)^2}\right\}$ by using convolution Theorem.

put $a=2$ in previous Problem

7. Find $L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\}$

Sol: $\frac{1}{p(p^2+4)^2} = \frac{p}{p^2(p^2+4)^2}$

Let $f(p) = \frac{1}{p^2}$, $g(p) = \frac{p}{(p^2+4)^2}$

Then

$L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2}\right\} = t = F(t)$

$L^{-1}\{g(p)\} = L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\} = \frac{t}{4}\sin 2t = G(t)$

By convolution Theorem

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t F(x)\,G(t-x)\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^2}\cdot\frac{p}{(p^2+4)^2}\right\} = \cancel{\int_0^t x \frac{x}{4} \sin 2(t-x)\,dx}$$

$$= \int_0^t F(t-x)\,G(x)\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\} = \int_0^t (t-x)\,\frac{x}{4}\sin 2x\,dx$$

$$= \frac{1}{4}\int_0^t [t\,x\sin 2x - x^2\sin 2x]\,dx$$

$$= \frac{t}{4}\int_0^t x\sin 2x\,dx - \frac{1}{4}\int_0^t x^2\sin 2x\,dx \longrightarrow (1)$$

$$\int_0^t x\sin 2x\,dx = x\left(-\frac{1}{2}\cos 2x\right) + \frac{1}{2}\int \cos 2x\,dx$$

$$= \left[-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 4x\right]_{x=0}^{t}$$

$$= -\frac{t}{2}\cos 2t + \frac{1}{4}\sin 4t \longrightarrow (2)$$

$$\int_0^t x^2\sin 2x\,dx = x^2\left[-\frac{1}{2}\cos 2x\right] + \frac{1}{2}\int 2x\cos 2x\,dx$$

$$= -\frac{x^2}{2}\cos 2x + \int x\cos 2x\,dx$$

$$= \left[-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x\right]_0^t \longrightarrow (3)$$

$$= -\frac{t^2}{2}\cos 2t + \frac{t}{2}\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}$$

using eq's (2) & (3) in eq (1), we get

$$L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\} = \frac{t}{4}\left[-\frac{t}{2}\cos 2t + \frac{1}{4}\sin 4t\right] - \frac{1}{4}\left[-\frac{t^2}{2}\cos 2t + \frac{t}{2}\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}\right]$$

$$= \frac{1}{4}\left\{-\frac{t^2}{2}\cos 2t + \frac{t}{4}\sin 2t + \frac{t^2}{2}\cos 2t - \frac{t}{2}\sin 2t + \frac{1}{4}\cos 2t - \frac{1}{4}\right\}$$

$$= \frac{1}{4}\left\{-\frac{t}{4}\sin 2t + \frac{1}{4}\cos 2t - \frac{1}{4}\right\}$$

$$= \frac{1}{16}\left\{\cos 2t - t\sin 2t - 1\right\}$$