Paper - VI

Linear Algebra

UNIT- I

vector spaces

vector spaces, General properties of vector spaces, Internal & external compositions, vector sub-spaces, algebra of subspaces, Linear sum of subspaces, linear combination of vectors, linear span, linear independence & dependence of vectors.

UNIT-II

Basis and Dimension

Basis of vector space, Finite dimensional vector space, Basis extension, coordinates, dimension of a vector space, dimension of a subspace, dimension of quotient space

UNIT-III

Linear Transformations

Linear Transformation, Linear operators, Sum & product of linear Transformation, Range & Null space of Linear Transformation, Rank, Nullity of linear trans-formation, Rank nullity theorem.

UNIT-IV

Matrices

Matrices, Elementary properties, Inverse matrices, Rank of a matrix, Linear equations, charac-teristic roots, characteristic vectors, cayley-Hamilton Theorem.

UNIT-V

Inner product spaces

Inner product spaces, Euclidean & unitary spaces, Norm of a vector, schwartz inequality, triangle inequality, parallelogram law, orthogonality, orthonormal set, complete orthonormal set, Gram-schmidt orthogonalization process, Bessel's Inequality, parseval's Identity.

UNIT-I

Vector spaces

Vector

The quantity which have both magnitude & direction is called a vector.

Ex:- velocity, acceleration, displacement & a triod of real numbers.

Scalar

The quantity which have only magnitude but no direction is called a scalar.

Ex:- mass, distance, speed, real numbers, all real numbers are scalars.

Scalar multiplication of a vector

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, ---, \alpha_n)$ be a vector in R^n. Let m be a scalar then '$m\alpha$' is also a vector whose direction is parallel to the direction of α & whose magnitude is 'm' ~~small~~ times to the magnitude of the vector α.

Here $m\alpha$ is called scalar multiple of α by m.

Addition of vectors

Let $\alpha = (\alpha_1, \alpha_2, ---\alpha_n)$ & $\beta = (\beta_1, \beta_2 --- \beta_n)$ are vectors in ~~R^n~~ R^n. The sum of vectors α, β is defined as $\alpha+\beta = (\alpha_1, \alpha_2, \alpha_3, ---, \alpha_n) + (\beta_1, \beta_2, \beta_3, --- \beta_n) = (\alpha_1+\beta_1, \alpha_2+\beta_2, \alpha_3+\beta_3, ---, \alpha_n+\beta_n)$

Internal composition

Let V be a non empty set of vectors. A mapping $f: V \times V \longrightarrow V$ is called an internal composition in V.

(or)

Let V be a non empty set of vectors. If $\alpha o \beta \in V$, $\forall \alpha, \beta \in V$ then 'o' is said to be an internal composition in the set V.

External composition

Let V be a nonempty set of vectors and F be a field of scalars. Then the mapping $f: F \times V \longrightarrow V$ is called an external composition in V over F.

(or)

Let V, F are nonempty sets if $a\circ\alpha \in V$, $\forall a \in F, \alpha \in V$ then '$\circ$' is said to be an external composition in V over F.

Here the resulting element $a\circ\alpha$ is an element in V.

Note

Internal composition is also called internal operation (or) binary operation.

External composition is called external operation (or) external binary composition.

Vector space

Let V be the nonempty set of vectors & F be the field of scalars & vector addition is internal composition in V & Scalar multiplication is external composition in V over F.

Then V is said to be a vector space over F if it satisfies the following conditions

(i) $(V,+)$ is an abelian group

(ii) Scalar multiplication property holds in V over F i.e., $a \in F, \alpha \in V \Rightarrow a\alpha \in V$.

The above two compositions satisfies the following properties.

(i) $a(\alpha+\beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha) = (a\alpha)b$

(iv) $1\cdot\alpha = \alpha$, $\forall \alpha, \beta \in V$, $a, b, 1 \in F$

Note

→ Instead of saying V is a vector space over F, we say that V(F) is a vector space

→ If R is a field of real numbers then V(R) is called the real vector space

→ If C is the field of complex numbers then $V(C)$ is called complex vector space.

Null space (or) zero space

The vector space having only one zero vector $\bar{0}$ is called the zero vector space.

Theorem-1

Let $V(F)$ be a vector space then (i) if $a,b \in F$ and $\alpha \in V$, $\alpha \neq \bar{0}$ then $a\alpha = b\alpha \Rightarrow a = b$

(ii) if $a \in F$, $a \neq 0$, and $\alpha, \beta \in V$ then $a\alpha = a\beta \Rightarrow \alpha = \beta$

proof:-

Let $V(F)$ be a vector space

(i) Let $a,b \in F$ and $\alpha \in V$, $\alpha \neq \bar{0}$

suppose $a\alpha = b\alpha$

$\Rightarrow a\alpha + (-b\alpha) = (b\alpha) + (-b\alpha)$

$\Rightarrow a\alpha - b\alpha = b\alpha - b\alpha$

$\Rightarrow (a-b)\alpha = \bar{0}, \alpha \neq \bar{0}$

$\Rightarrow a - b = 0$

$\Rightarrow a = b$

$\therefore a\alpha = b\alpha \Rightarrow a = b$

(ii) Let $a \in F$ and $a \neq 0$

Let $\alpha, \beta \in V$

suppose $a\alpha = a\beta$

$\Rightarrow a\alpha - a\beta = \bar{0}$

$\Rightarrow a(\alpha - \beta) = 0,\ a \neq 0$

$\Rightarrow \alpha - \beta = 0$

$\Rightarrow \alpha = \beta$

$\therefore a\alpha = a\beta, a \neq 0 \Rightarrow \alpha = \beta$

Examples of vector space

1) The set C_n of all n tuples of complex numbers forms a vector space with respect to addition & scalar multiplication of complex numbers defined as follows.

$$1)\ \alpha + \beta = (\alpha_1, \alpha_2, ---, \alpha_n) + (\beta_1, \beta_2, ---, \beta_n)$$
$$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ---, \alpha_n + \beta_n),\ \forall \alpha = (\alpha_1, \alpha_2, ---, \alpha_n),$$
$$\beta = (\beta_1, \beta_2, --- \beta_n) \in C_n$$

$$2)\ m\alpha = (m\alpha_1, m\alpha_2, ---, m\alpha_n),\ \forall m \in C,\ \alpha \in C_n$$

2) The set of all real valued functions defined in the $(0,1)$ is a vector space over the field of real numbers.

w.r the operations of addition & Scalar multiplication defined as follows.

1) $(f+g)(x) = f(x)+g(x)$

2) $(mf)(x) = m f(x)$, where $0<x<1$

3) Let v is the set of all 'mxn' matrices with real entries & R is the field of real numbers.

Addition of matrices is the internal operation & multiplication of a matric by a real number is an external composition in V.

Then V is a vector space over the field R.

1) Let V be the set of all pairs (a,b) of real numbers & R be the field of real numbers show that the operation $(a_1, b_1)+(a_2, b_2) = (a_1+a_2, 0)$ & $c(a_1, b_1) = (ca_1, b_1)$ show that V(R) is not a vector space

Let V be the set of all pairs (a,b) of real numbers

i.e $V = \{(a,b) / a, b \in R\}$

and R be the field of real numbers

Given that: The operators in V defined as

$$(a_1, b_1)+(a_2, b_2) = (a_1+a_2, 0)$$

and

$$c(a_1, b_1) = (ca_1, b_1), \forall (a_1, b_1), (a_2, b_2) \in V \text{ and } c \in R$$

Let (x,y) be the identity element in V

Then $(a+b)+(x+y) = (a,b)$

$\Rightarrow (a+x, 0) = (a,b)$

$\Rightarrow b = 0$

∴ This is a contradiction

Hence addition of vectors in V is not an internal composition [i.e. identity does not exist in V]

∴ V(R) is not a vector space

Imp Vector subspace

Let $V(F)$ be a vector space & $W \subseteq V$. Then W is said to be a subspace of V if W itself is a vector space over the field F with the same operations of vector addition & scalar multiplication in V

Note

→ Let $V(F)$ be a vector space then the zero space $\{\bar{0}\} \subseteq V$ & $V \subseteq V$

$\therefore$ $\{\bar{0}\}$ & V are trivial subspaces of the vector space $V(F)$

→ If $W(F)$ is a subspace of the vector space $V(F)$ then $(W,+)$ is a subgroup of $(V,+)$

Theorem - 2

VV Imp 10M Let $V(F)$ be a vector space & $W \subseteq V$. Then the necessary & sufficient conditions for W to be a subspace of V are

i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Proof:-

Let ~~&~~ $V(F)$ be a vector space and

$W \subseteq V$ [i.e W is a subset of V]

we have to prove that

W is a subspace of $V(F) \Leftrightarrow W$ satisfies two conditions

1) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

2) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Suppose W is a subspace of $V(F)$

i.e W is itself a vector space over F with respect to vector addition & scalar multiplication defined in V.

$\because$ $W(F)$ is an vector space

$\Rightarrow (W,+)$ is an abelian group

Let $\alpha, \beta \in W$

$\Rightarrow \alpha, -\beta \in W$ [by inverse axiom in W w.r to +]

$\Rightarrow \alpha+(-\beta) \in W$ [by closure axiom in W w.r to +]

$\Rightarrow \alpha-\beta \in W$

$\boxed{\therefore \alpha, \beta \in W \Rightarrow \alpha-\beta \in W}$

Since W(F) is a vector space

$\Rightarrow$ W satisfies scalar multiplication property over F

$\boxed{\text{i.e } a \in F, \alpha \in W \Rightarrow a\alpha \in W}$

Hence both the conditions holds in R

Converse

Suppose W satisfies the following two conditions

i) $\alpha \in W, \beta \in W \Rightarrow \alpha-\beta \in W \rightarrow (1)$

ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W \rightarrow (2)$

we have to prove that

W(F) is a subspace of V(F)

It is equivalent to prove that

W is itself a vectorspace over F

From eq(1): $\alpha, \alpha \in W \Rightarrow \alpha-\alpha \in W$

$\Rightarrow \bar{0} \in W$

Identity element $(\bar{0})$ w.r to addition lies in W

From eq(1): $\bar{0} \in W, \alpha \in W \Rightarrow \bar{0}-\alpha \in W$

$\Rightarrow -\alpha \in W$

$\therefore$ every element of W has additive inverse in W

Let $\alpha, \beta \in W \Rightarrow \alpha, -\beta \in W$

From eq(1): $\alpha, -\beta \in W \Rightarrow \alpha-(-\beta) \in W$

$\Rightarrow \alpha+\beta \in W$

$\therefore$ W satisfies closure axiom w.r to vector addition

Since $W \subseteq V$

$\Rightarrow$ every element of W is an element of V &

The elements of V satisfies associative & commutative axioms w.r to vector addition

$\therefore$ elements of W also satisfies associative & commutative axioms w.r to vector addition

Hence $\boxed{(W,+) \text{ is an abelian group}}$

From eq② : W satisfies scalar multiplication property over F

i.e $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Trivially, the remaining properties of vector space holds in W

$\therefore$ W is itself a vector space over F & $W \subseteq V$

$\therefore$ $W(F)$ is a subspace of $V(F)$

This completes the proof

Theorem-3

Imp 10M Let $V(F)$ be a vector space & W is a nonempty subset of V. The necessary & sufficient condition for W to be a subspace of V is $a,b \in F,\ \alpha,\beta \in W \Rightarrow a\alpha + b\beta \in W$

proof:-

Let $V(F)$ be a vector space and

W be a nonempty subset of V

we have to prove that

$W(F)$ is a subspace of $V(F) \Leftrightarrow W$ satisfies the condition

$a,b \in F,\ \alpha,\beta \in W \Rightarrow a\alpha + b\beta \in W$

suppose $W(F)$ is a subspace of $V(F)$

$\Rightarrow W$ is itself a vector space over F w.r to vector addition & scalar multiplication same as defined in V

$\Rightarrow (W,+)$ is an abelian group and

W satisfies the scalar multiplication property over F.

$\because a \in F, \alpha \in W \Rightarrow a\alpha \in W$ [by scalar multiplication property in W]

$\because b \in F, \beta \in W \Rightarrow b\beta \in W$ [by "]

$\because a\alpha \in W, b\beta \in W \Rightarrow a\alpha + b\beta \in W$ [by closure axiom in W w.r to $+$)

$\therefore W$ satisfies the condition:

$a,b \in F,\ \alpha,\beta \in W \Rightarrow a\alpha + b\beta \in W$

Converse

Suppose W satisfies the condition

$a, b \in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W \longrightarrow ①$

we have to prove that $W(F)$ is a subspace of $V(F)$

By taking $a=1, b=-1$, then eq① $\Rightarrow$

$\alpha, \beta \in W \Rightarrow \alpha - \beta \in W \longrightarrow ②$

By taking $b=0$, then eq② $\Rightarrow$

$a \in F, \alpha \in W \Rightarrow a\alpha \in W \longrightarrow ③$

Here eq's ② & ③ are necessary & sufficient conditions for a nonempty set 'W' to be a subspace of V

$\therefore W(F)$ is a subspace of $V(F)$

Hence proved

Imp Theorem-4

A nonempty set W is a subset of the vector space $V(F)$ then W is a subspace of $V(F) \Leftrightarrow a \in F, \alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$

proof:-

Let $V(F)$ be a vector space and

W is a nonempty subset of V

we have to prove that

$W(F)$ is a subspace of $V(F) \Leftrightarrow W$ satisfies the condition:

$a \in F, \alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$

Suppose $W(F)$ is a subspace of $V(F)$

$\Rightarrow W$ is itself a vector space over F w.r to vector addition & scalar multiplication defined same as in $V(F)$

$\Rightarrow (W, +)$ is an abelian group and

W satisfies the scalar multiplication property over F

$\because a \in F, \alpha \in W \Rightarrow a\alpha \in W$ [by S.M.P in W]

$\because a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$ [by closure axiom in W w.r to +]

$\therefore W$ satisfies the condition

$a\in F,\ \alpha,\beta\in W \Rightarrow a\alpha+\beta\in W$

Converse

Suppose W satisfies the condition

$a\in F,\ \alpha,\beta\in W \Rightarrow a\alpha+\beta\in W \longrightarrow ①$

we have to prove that $W(F)$ is a subspace of $V(F)$

By taking $a=1$ in eq①, we have

$\alpha,\beta\in W \Rightarrow \alpha+\beta\in W$

$\therefore$ W is closed w.r to vector addition

By taking $a=-1$ & $\alpha,\alpha\in W$, eq① $\Rightarrow$

$-\alpha+\alpha\in W$

$\Rightarrow \bar{0}\in W$

$\therefore$ Identity element $(\bar{0})$ exists in W w.r to vector addition

By taking $a=-1$, &

$\alpha,\bar{0}\in W$, Then eq① $\Rightarrow$

$-\alpha+\bar{0}\in W \Rightarrow -\alpha\in W$

i.e $\alpha\in W \Rightarrow -\alpha\in W$

$\therefore$ every element of W has inverse in W w.r to vector addition

Since $W\subseteq V$

$\Rightarrow$ every element of W is an element of V and the elements of V satisfies associative & commutative axioms w.r to vector addition

$\therefore$ The elements of W also satisfies associative axiom & commutative axiom w.r to vector addition

$\therefore$ $\boxed{(W,+) \text{ is an abelian group}}$

Now eq① $\Rightarrow$ $a\in F,\ \alpha,\bar{0}\in W \Rightarrow a\alpha+\bar{0}\in W$

$\Rightarrow a\alpha\in W$

$\therefore a\in F,\ \alpha\in W \Rightarrow a\alpha\in W$

$\therefore$ W satisfies scalar multiplication property over F

Trivially, the remaining properties of vector space holds in W

$\therefore$ W is itself a vector space over F and $W \subseteq V$

$\therefore$ W(F) is a subspace of V(F)

Hence proved

Note:-

A nonempty set W is a subspace of vector space V(F)

$\Leftrightarrow a\alpha + b\beta \in W,\ \forall\ \alpha, \beta \in W,\ a, b \in F$

5M Imp method 1) show that the set W of all triods $(x, y, 0)$, where $x, y \in F$ is a subspace of $V_3(F)$.

Given that

$V_3(F)$ be the vector space $(V_3 = R^3)$

and $W = \{(x, y, 0) / x, y \in F\}$

clearly $W \subseteq V_3$

Let $\alpha, \beta \in W$

$\Rightarrow \alpha = (x_1, y_1, 0)$

$\beta = (x_2, y_2, 0)$, where $x_1, x_2, y_1, y_2 \in F$

Take $a, b \in F$

consider

$a\alpha + b\beta = a(x_1, y_1, 0) + b(x_2, y_2, 0)$

$= (ax_1, ay_1, 0) + (bx_2, by_2, 0)$

$= (ax_1 + bx_2,\ ay_1 + by_2,\ 0)$

$= (c, d, 0)$, where $c = ax_1 + bx_2 \in F$

$d = ay_1 + by_2 \in F$

$\therefore a\alpha + b\beta = (c, d, 0)$, where $c, d \in F$

$\Rightarrow (c, d, 0) \in W$

$\Rightarrow a\alpha + b\beta \in W$

$\therefore a, b \in F,\ \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\therefore$ W is a subspace of $V_3(F) = (R^3(R))$

2) Let R be the field of real numbers then show that the set of triods $\{(x, 2y, 3z) / x, y, z \in R\}$ form the subspace of $R^3(R)$

Given that

$R^3(R)$ be the vector space

and $W = \{(x, 2y, 3z) / x, y, z \in R\}$

clearly $W \subseteq R^3$

Let $\alpha, \beta \in W$

$\Rightarrow \alpha = (x_1, 2y_1, 3z_1)$

$\beta = (x_2, 2y_2, 3z_2)$, where $x_1, y_1, z_1, x_2, y_2, z_2 \in R$

Take $a, b \in R$

consider

$a\alpha + b\beta = a(x_1, 2y_1, 3z_1) + b(x_2, 2y_2, 3z_2)$

$= (ax_1, 2ay_1, 3az_1) + (bx_2, 2by_2, 3bz_2)$

$= (ax_1 + bx_2, 2ay_1 + 2by_2, 3az_1 + 3bz_2)$

$= (ax_1 + bx_2, 2(ay_1 + by_2), 3(az_1 + bz_2))$

$= (p, 2q, 3r)$, where $p = ax_1 + bx_2 \in R$

$q = ay_1 + by_2 \in R$, $R = az_1 + bz_2 \in R$

$\therefore a\alpha + b\beta = (p, 2q, 3r)$, where $p, q, r \in R$

$\Rightarrow (p, 2q, 3r) \in W$

$\Rightarrow a\alpha + b\beta \in W$

$\therefore a, b \in R$, $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\therefore W$ is a subspace of $V_3(F) = R^3(R)$

3) Let R be the field of real numbers then show that the set of triads $\{(x, x, x) / x \in R\}$ forms a subspace of $R^3(R)$

Given that

$R^3(R)$ be the vector space

and $W = \{(x, x, x) / x \in R\}$

clearly $W \subseteq R^3$

Let $\alpha, \beta \in W$

$\Rightarrow \alpha = (x, x, x)$

$\beta = (y, y, y)$ where $x, y \in R$

Take $a, b \in R$

consider

$a\alpha + b\beta = a(x, x, x) + b(y, y, y)$

$= (ax, ax, ax) + (by, by, by)$

$= (ax+by, ax+by, ax+by)$

$= (c, c, c)$; $c = ax+by \in R$

$\therefore a\alpha + b\beta = (c, c, c)$; $c \in R$

$\Rightarrow (c, c, c) \in W$

$\Rightarrow a\alpha + b\beta \in W$

$\therefore a, b \in R$, $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

$\therefore W$ is a subspace of $V_3(F) = R^3(R)$

4) Let $V = R^3 = \{(x, y, z) / x, y, z \in R\}$ & W be the set of triods x, y, z such that $x - 3y + 4z = 0$. Show that W is a subspace of $V(R)$

Given that

$V_3 = R^3 = \{(x, y, z) \mid x, y, z \in R\}$ and

$W = \{(x, y, z) / x - 3y + 4z = 0\}$

~~Let~~ clearly, $W \subseteq R^3$

Let $\alpha, \beta \in W$, Then

$\alpha = (x, y, z) \Rightarrow x - 3y + 4z = 0 \longrightarrow (1)$

$\beta = (p, q, r) \Rightarrow p - 3q + 4r = 0 \longrightarrow (2)$

Now for $a, b \in R$,

~~$a\alpha + b\beta = a(x - 3y + 4z) + b(p - 3q + 4r)$~~

~~$= (ax - 3ay + 4az) + (bp - 3bq + 4br)$~~

~~$= ax + bp,\ aq$~~

$= a(x, y, z) + b(p, q, r)$

$= (ax, ay, az) + (bp, bq, br)$

$= (ax+bp, ay+bq, az+br) \longrightarrow (3)$

Consider

$(ax+bp) - 3(ay+bq) + 4(az+br) = ax+bp-3ay-3bq+4az+4br$

$= a[x-3y+4z] + b[p-3q+4r]$

$= a(0) + b(0)$ [by eq's ① & ②]

$= 0$

$\therefore (ax+bp) - 3(ay+bq) + 4(az+br) = 0$

$\Rightarrow (ax+bp, ay+bq, az+br) \in W$

$\Rightarrow a\alpha+b\beta \in W$ (by eq ③)

$\therefore a,b \in R, \alpha,\beta \in W \Rightarrow a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $R^3(R)$

5) prove that the set of solutions (x,y,z) of the equation $x+y+2z=0$ is a subspace of $R^3(R)$

Given that

$V_3 = R^3 = \{(x,y,z) / x,y,z \in R\}$ and

$W = \{(x,y,z) / x+y+2z=0\}$

Clearly $W \subseteq R^3$

Let $\alpha,\beta \in W$, Then

$\alpha = (x,y,z) \Rightarrow x+y+2z=0 \rightarrow$ ①

$\beta = (p,q,r) \Rightarrow p+q+2r=0 \rightarrow$ ②

Now for $a,b \in R$,

$a\alpha+b\beta = a(x,y,z)+b(p,q,r)$

$=(ax,ay,az)+(bp,bq,br)$

$=(ax+bp, ay+bq, az+br) \rightarrow$ ③

consider

$(ax+bp)+(ay+bq)+2(az+br) = ax+bp+ay+bq+2az+2br$

$= a(x+y+2z)+b(p+q+2r)$

$= a(0)+b(0) = 0$

$\therefore (ax+bp)+(ay+bq)+2(az+br) = 0$

$\Rightarrow (ax+bp, ay+bq, az+br) \in W$

$\Rightarrow a\alpha+b\beta \in W$ [by eq③]

$\therefore a,b \in R, \alpha,\beta \in W \Rightarrow a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $R^3(R)$

6) Let p, q, r be the fixed elements show that the set W of all triads x, y, z of elements of F such that $px+qy+rz=0$ is a vector subspace of $V_3(F)$

V.V Imp 5M

Let p, q, r be the fixed elements of a field F

Given that $W = \{(x,y,z) / px+qy+rz=0\}$

Clearly, $W \subseteq V_3(F)$

Let $\alpha, \beta \in W$, Then

$\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$ such that

$px_1+qy_1+rz_1=0 \rightarrow ①$

$px_2+qy_2+rz_2=0 \rightarrow ②$

Take $a, b \in F$, $\alpha, \beta \in W$, consider

$a\alpha+b\beta = a(x_1, y_1, z_1)+b(x_2, y_2, z_2)$

$= (ax_1, ay_1, az_1)+(bx_2, by_2, bz_2)$

$= (ax_1+bx_2, ay_1+by_2, az_1+bz_2)$

Now

$p[ax_1+bx_2]+q[ay_1+by_2]+r[az_1+bz_2] = pax_1+pbx_2+qay_1+qby_2+raz_1+rbz_2$

$= a(px_1+qy_1+rz_1)+b(px_2+qy_2+rz_2)$

$= a(0)+b(0)=0$

$\therefore p[ax_1+bx_2]+q[ay_1+by_2]+r[az_1+bz_2]=0$

$\Rightarrow (ax_1+bx_2, ay_1+by_2, az_1+bz_2) \in W$

$\Rightarrow a\alpha+b\beta \in W$

$\therefore a, b \in F$, $\alpha, \beta \in W \Rightarrow a\alpha+b\beta \in W$

$\therefore$ W is a subspace of $V_3(F)$

7) Let R be the field of real numbers & $W = \{(x,y,z) / x,y,z$ are rational numbers$\}$. Is W a subspace of $V_3(R)$

Let R be the field of real numbers and

$W = \{(x,y,z) / x,y,z$ are rational numbers$\}$

Let $\alpha = (2,3,4) \in W$

Let $m = \sqrt{5} \in R$ be a real number

$m\alpha = \sqrt{5}(2,3,4)$

$= (2\sqrt{5}, 3\sqrt{5}, 4\sqrt{5})$

$\because 2\sqrt{5}, 3\sqrt{5}$ are irrational numbers

Then $(2\sqrt{5}, 3\sqrt{5}, 4\sqrt{5}) \notin W$

i.e $m\alpha \notin W$

i.e W not satisfies scalar multiplication

$\therefore$ W is not a subspace

8) Show that the subset W defined as $W = \{(a,b,c) / a^2+b^2+c^2 \le 1\}$ is not a subspace of $V_3(R)$

Let R be the field of real numbers and

$W = \{(a,b,c) / a^2+b^2+c^2 \le 1\}$

Let $\alpha = (1,0,0) \in W$, $\beta = (0,1,0) \in W$

Now

$\alpha+\beta = (1,0,0)+(0,1,0)$

$= (1,1,0)$

Now $1^2+1^2+0^2 \nleq 1$

$\Rightarrow \alpha+\beta \notin W$

$\therefore \alpha, \beta \in W \Rightarrow \alpha+\beta \notin W$

$\therefore$ W not satisfies the closure axiom w.r to vector addition

$\therefore$ W is not a subspace of $V_3(R)$

9) Show that the set W of the elements of the vector space $V_3(R)$ of the form $W = \{(x+2y, y, -x+3y) / x,y \in R\}$ is a subspace of $V_3(R)$

Let R be the field of real numbers and

$W = \{(x+2y, y, -x+3y) / x,y \in R\}$

Let $\alpha, \beta \in W$, $\alpha = (x_1+2y_1, y_1, -x_1+3y_1)$

$\beta = (x_2+2y_2, y_2, -x_2+3y_2)$

Take $a, b \in R$ and consider

$a\alpha + b\beta = a(x_1+2y_1, y_1, -x_1+3y_1) + b(x_2+2y_2, y_2, -x_2+3y_2)$

$= (ax_1+2ay_1, ay_1, -ax_1+3ay_1) + (bx_2+2by_2, by_2, -bx_2+3by_2)$

$= (ax_1+2ay_1+bx_2+2by_2, ay_1+by_2, -ax_1+3ay_1-bx_2+3by_2)$

$= [(ax_1+bx_2)+2(ay_1+by_2), ay_1+by_2, -(ax_1+bx_2)+3(ay_1+by_2)]$;

$(\because p = ax_1+bx_2, q = ay_1+by_2$

$= (p+2q, q, -p+3q) \in W$

$\therefore a\alpha+b\beta \in W, \forall a,b \in R, \alpha,\beta \in W$

$\therefore W$ is a sub space of $V_3(R)$

10) Let v be the vector space of all polynomials in indeterminate. x over the field F. Let ω be a subset of v consisting of all polynomials of degree $\leq n$. Then show that ω is a subspace of V(F).

Given that

V be the vector space of all polynomials in the variable x over the field F.

W be the set of polynomials in x of degree $\leq n$

Clearly W is subset are equivalent to v $[W \subseteq V]$

Let $\alpha, \beta \in W$

i-e α, β are polynomials in x of degree $\leq n$

Let $a,b \in F$; Then $a\alpha, b\beta$ are also polynomials in x of degree $\leq n$

Then $a\alpha+b\beta$ is also a polynomial of degree $\leq n$

$\therefore a\alpha+b\beta \in W$

$\therefore$ W is a subspace of v over F

Theorem-5

VV Imp The intersection of any two subspaces W_1 & W_2 of a vector space V(F) is also a subspace

proof:-

Let V(F) be a vector space and W_1, W_2 are two subspaces of V(F)

We have to prove that

$W_1 \cap W_2$ is also a subspace of V(F)

Since W_1, W_2 are subspaces of V(F)

$\Rightarrow \bar{0} \in W_1$ & $\bar{0} \in W_2$

$\Rightarrow \bar{0} \in W_1 \cap W_2$

$\Rightarrow W_1 \cap W_2 \neq \phi$

Let $\alpha, \beta \in W_1 \cap W_2$

$\Rightarrow \alpha, \beta \in W_1$ & $\alpha, \beta \in W_2$

Let $a, b \in F$ & $\alpha, \beta \in W_1$

$\Rightarrow a\alpha + b\beta \in W_1 \longrightarrow ①$ [because W_1 is a subspace of $V(F)$]

Take $a, b \in F$ & $\alpha, \beta \in W_2$

$\Rightarrow a\alpha + b\beta \in W_2 \longrightarrow ②$ [because W_2 is a subspace of $V(F)$]

From eq's ① & ②, we have

$a\alpha + b\beta \in W_1 \cap W_2$

$\therefore a, b \in F$, $\alpha, \beta \in W_1 \cap W_2 \Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

$\therefore W_1 \cap W_2$ is a subspace of $V(F)$

Hence proved

Theorem-6

The intersection of any family of subspaces of a vector space is also a subspace of $V(F)$

proof:-

Let $V(F)$ be a vector space and

Let $W_1, W_2, \ldots W_n$ are 'n' subspaces of $V(F)$

we have to prove that

$W_1 \cap W_2 \cap \ldots \cap W_n = \bigcap_{i=1}^{n} W_i$ is a subspace of $V(F)$

Since $W_1, W_2, \ldots, W_n$ are subspaces of $V(F)$

$\Rightarrow \bar{0} \in W_1$, $\bar{0} \in W_2$, ..., $\bar{0} \in W_n$

$\Rightarrow \bar{0} \in \bigcap_{i=1}^{n} W_i$

$\Rightarrow \bigcap_{i=1}^{n} W_i \neq \emptyset$

Let $\alpha, \beta \in \bigcap_{i=1}^{n} W_i$

$\Rightarrow \alpha, \beta \in W_1$, $\alpha, \beta \in W_2$, ..., $\alpha, \beta \in W_n$

Let $a, b \in F$ & $\alpha, \beta \in W_i$, $\forall i = 1, 2, \ldots n$

$\Rightarrow a\alpha + b\beta \in W_i \longrightarrow ①$ $\forall i = 1, 2, \ldots n$ [because W_i is a subspace of $V(F)$]

$\Rightarrow a\alpha + b\beta \in \bigcap_{i=1}^{n} W_i$

$\therefore a, b \in F,\ \alpha, \beta \in \bigcap_{i=1}^{n} W_i \Rightarrow a\alpha + b\beta \in \bigcap_{i=1}^{n} W_i$

$\therefore \bigcap_{i=1}^{n} W_i$ is a subspace of $V(F)$

Hence proved

Result / Is the union of two subspaces is subspace or not.

Imp. The union of two subspaces of $V(F)$ need not be a subspace of $V(F)$

Ex:- Let $V(F)$ be a vector space

Let $W_1 = \{(0, y, 0) / y \in F\}$ and

$W_2 = \{(0, 0, z) / z \in F\}$

clearly W_1, W_2 are subspaces of $V(F)$

Now

$W_1 \cup W_2 = \{(0, y, 0) \cup (0, 0, z) / y, z \in F\}$

Let $\alpha = (0, y, 0) \in W_1$, $\beta = (0, 0, z) \in W_2$

$\alpha + \beta = (0, y, 0) + (0, 0, z)$

$= (0, y, z)$

$\therefore \alpha + \beta = (0, y, z) \notin W_1 \cup W_2$, because

neither $(0, y, z) \in W_1$ nor $(0, y, z) \in W_2$

i.e $(0, y, z) \notin W_1$, $(0, y, z) \notin W_2$

$\therefore W_1 \cup W_2$ does not satisfy closure axiom w.r to vector addition

$\therefore W_1 \cup W_2$ is not a subspace of $V(F)$

Hence, the union of two subspaces of a vector space need not be a subspace

Imp 10M Theorem-7

The union of two subspaces is a subspace $\Leftrightarrow$ one contained in the other.

proof:-

Let $V(F)$ be a vector space and

W_1, W_2 are two subspaces of $V(F)$

we have to prove that

$W_1 \cup W_2$ is a subspace of $V(F) \Leftrightarrow W_1 \subset W_2$ (or) $W_2 \subset W_1$

Suppose $W_1 \cup W_2$ is a subspace of $V(F)$

we have to prove that $W_1 \subset W_2$ (or) $W_2 \subset W_1$

If possible assume that

$W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

$\because W_1 \not\subset W_2 \Rightarrow \exists$ an element $a \in W_1$ but $a \notin W_2 \longrightarrow$ (1)

$\because W_2 \not\subset W_1 \Rightarrow \exists$ an element $b \in W_2$ but $b \notin W_1 \longrightarrow$ (2)

$\because a \in W_1, b \in W_2 \Rightarrow a, b \in W_1 \cup W_2$

$\because a, b \in W_1 \cup W_2$ & $W_1 \cup W_2$ is a subspace of $V(F)$

$\Rightarrow a-b \in W_1 \cup W_2$

$\Rightarrow a-b \in W_1$ (or) $a-b \in W_2$

Take $a-b \in W_1$ & $a \in W_1$ & W_1 is a subspace

$\Rightarrow a-(a-b) \in W_1$

$\Rightarrow b \in W_1 \longrightarrow$ (3)

Again take $a-b \in W_2$, $b \in W_2$

$\Rightarrow a-b+b \in W_2$ [by closure axiom in W_2 w.r to +]

$\Rightarrow a \in W_2 \longrightarrow$ (4)

clearly, eq (4), (3) contradicts (1), (2) respectively

$\therefore$ our assumption $W_1 \not\subset W_2$ & $W_2 \not\subset W_1$ is wrong

Hence $W_1 \subset W_2$ (or) $W_2 \subset W_1$

Converse

Suppose $W_1 \subset W_2$ (or) $W_2 \subset W_1$

Then

$$W_1 \cup W_2 = \begin{cases} W_2 & \text{if } W_1 \subset W_2 \\ W_1 & \text{if } W_2 \subset W_1 \end{cases}$$

In both the cases, $W_1 \cup W_2$ is a subspace of $V(F)$

Hence, $W_1 \cup W_2$ is a subspace of $V(F) \Leftrightarrow W_1 \subset W_2$ (or) $W_2 \subset W_1$

Linear sum of two subspaces

Let $V(F)$ be a vector space & W_1, W_2 are any two subspaces of $V(F)$. Then the linear sum of the subspaces is denoted by $W_1 + W_2$ & defined as the set of all elements $\alpha_1 + \alpha_2$, where $\alpha_1 \in W_1$, $\alpha_2 \in W_2$

i.e $W_1 + W_2 = \{\alpha_1 + \alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\}$

Theorem - 8

If W_1 & W_2 are any two subspaces of a vector space $V(F)$ then

(i) W_1+W_2 is a subspace of $V(F)$

(ii) $W_1 \subseteq W_1+W_2$ & $W_2 \subseteq W_1+W_2$

proof:-

Let $V(F)$ be a vector space &

W_1, W_2 are any two subspaces of $V(F)$

The linear sum of W_1 & W_2 is denoted by W_1+W_2 &

defined as $W_1+W_2 = \{\alpha_1+\alpha_2 / \alpha_1 \in W_1, \alpha_2 \in W_2\}$

Trivially $W_1+W_2 \neq \emptyset$ [$\because \bar{0} \in W_1+W_2$]

To prove that W_1+W_2 is a subspace of $V(F)$

Let $\alpha, \beta \in W_1+W_2$, Then

$\alpha = \alpha_1+\alpha_2$ & $\beta = \beta_1+\beta_2$, where $\alpha_1, \beta_1 \in W_1$ & $\alpha_2, \beta_2 \in W_2$

Take $a, b \in F$

consider

$a\alpha+b\beta = a(\alpha_1+\alpha_2)+b(\beta_1+\beta_2)$

$= a\alpha_1+a\alpha_2+b\beta_1+b\beta_2$

$= (a\alpha_1+b\beta_1)+(a\alpha_2+b\beta_2) \rightarrow (1)$

$\because a, b \in F$, $\alpha_1, \beta_1 \in W_1$ and W_1 is a subspace

$\Rightarrow a\alpha_1+b\beta_1 \in W_1 \rightarrow (2)$

$\because a, b \in F$, $\alpha_2, \beta_2 \in W_2$ and W_2 is a subspace

$\Rightarrow a\alpha_2+b\beta_2 \in W_2 \rightarrow (3)$

From eq (2) & (3), we have

$(a\alpha_1+b\beta_1)+(a\alpha_2+b\beta_2) \in W_1+W_2$

$\Rightarrow a\alpha+b\beta \in W_1+W_2$

$\therefore a, b \in F$ & $\alpha, \beta \in W_1+W_2 \Rightarrow a\alpha+b\beta \in W_1+W_2$

$\therefore W_1+W_2$ is a subspace of $V(F)$

To prove that $W_1 \subseteq W_1+W_2$ and $W_2 \subseteq W_1+W_2$

$\because W_1, W_2$ are subspaces of $V(F)$

$\Rightarrow \bar{0} \in W_1, \bar{0} \in W_2$

$\because \alpha \in W_1, \bar{0} \in W_2 \Rightarrow \alpha+\bar{0} \in W_1+W_2$

$\Rightarrow \alpha \in W_1+W_2$

$\therefore \alpha \in W_1 \Rightarrow \alpha \in W_1+W_2$

$\therefore W_1 \subseteq W_1+W_2$

lly $W_2 \subseteq W_1+W_2$

Imp **Linear combination of vectors**

Let $V(F)$ be a vector space. If $\alpha_1, \alpha_2, \dots \alpha_n \in V$ then any vector $\gamma = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$, where $a_1, a_2, \dots a_n \in F$, is called linear combination of vectors $\alpha_1, \alpha_2, \dots \alpha_n$

Note

Here the vector $\gamma \in V(F)$

Imp **Linear span of a set**

Let S be a nonempty subset of a vector space $V(F)$. The linear span of S is denoted by $L(S)$ & defined as the set of all possible linear combinations of elements of all possible finite subsets of S i.e $L(S) = \{\gamma / \gamma = \sum_i a_i\alpha_i,\ a_i \in F, \alpha_i \in S\}$

Note

1) S may be a finite set but $L(S)$ is infinite set

2) $L(S)$ is said to be generated by S or spanned by S

3) If S is an empty set then $L(S)$ is defined as $L(S) = \{0\}$

4) Always $L(S) \neq \emptyset$

5) Always $S \subseteq L(S)$

Theorem-9

5M Imp The Linear span $L(S)$ of any subset S of a vector space $V(F)$ is a subspace of $V(F)$

proof:-

Let $V(F)$ be a vector space and S be a nonempty subset of $V(F)$

The linear span of S is denoted by $L(S)$ & defined as

$$L(S) = \{\gamma / \gamma = \sum_i a_i\alpha_i,\ a_i \in F, \alpha_i \in S\}$$

clearly, $L(S) \subset V(F)$

we have to prove that

$L(S)$ is a subspace of $V(F)$

Let $\alpha, \beta \in L(S)$

$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$

$\beta = b_1\beta_1 + b_2\beta_2 + \dots + b_n\beta_n$, where

$\alpha_i, \beta_i \in S, \forall\ b_i, a_i \in F, \forall i = 1, 2, 3 \dots n$

For $a, b \in F$, consider

$a\alpha + b\beta = a[a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n] + b[b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n]$

$= (aa_1)\alpha_1 + (aa_2)\alpha_2 + \cdots + (aa_n)\alpha_n + (bb_1)\beta_1 + (bb_2)\beta_2 + \cdots + (bb_n)\beta_n$

$= a_1'\alpha_1 + a_2'\alpha_2 + \cdots + a_n'\alpha_n + b_1'\beta_1 + b_2'\beta_2 + \cdots + b_n'\beta_n$

= Linear combination of $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots \beta_n$

= Linear combination of elements of $S \in L(S)$

$\therefore a\alpha + b\beta \in L(S)$

$\therefore a, b \in F,\ \alpha, \beta \in L(S) \Rightarrow a\alpha + b\beta \in L(S)$

$\therefore L(S)$ is a subspace of $V(F)$

Note

Let S be a nonempty subset of the vector space $V(F)$. Then the linear span $L(S)$ is the intersection of all subspaces of V which contains S.

Theorem-10

10M Imp. If W_1 & W_2 are any two subspaces of a vector space $V(F)$ then $L(W_1 \cup W_2) = W_1 + W_2$

proof:-

Let $V(F)$ be a vector space and W_1, W_2 are any two subspaces of $V(F)$

We have to prove that

$L(W_1 \cup W_2) = W_1 + W_2$

Since W_1, W_2 are two subspaces of $V(F)$

$\Rightarrow \bar{0} \in W_1$ & $\bar{0} \in W_2$

Let $\alpha_1 \in W_1$ & $\alpha_2 \in W_2$

$\because \alpha_1 \in W_1, \bar{0} \in W_2 \Rightarrow \alpha_1 + \bar{0} \in W_1 + W_2$

$\Rightarrow \alpha_1 \in W_1 + W_2$

$\therefore \alpha_1 \in W_1 \Rightarrow \alpha_1 \in W_1 + W_2$

$\therefore W_1 \subset W_1 + W_2 \rightarrow (1)$

$\because \alpha_2 \in W_2, \bar{0} \in W_1 \Rightarrow \bar{0} + \alpha_2 \in W_1 + W_2$

$\Rightarrow \alpha_2 \in W_1 + W_2$

$\therefore \alpha_2 \in W_2 \Rightarrow \alpha_2 \in W_1 + W_2$

$\therefore W_2 \subset W_1 + W_2 \longrightarrow (2)$

$\therefore$ From eq's ① & ②, we have

$W_1 \cup W_2 \subset W_1 + W_2$

$\Rightarrow L(W_1 \cup W_2) \subset W_1 + W_2 \longrightarrow ③$

$\because \alpha_1 \in W_1, \alpha_2 \in W_2 \Rightarrow \alpha_1 + \alpha_2 \in W_1 + W_2$

$\therefore \alpha_1 + \alpha_2 \in W_1 + W_2$

$\alpha_1 \in W_1, \alpha_2 \in W_2$

$\Rightarrow \alpha_1, \alpha_2 \in W_1 \cup W_2$

Now

$\alpha_1 + \alpha_2$ = Linear combination of $\alpha_1, \alpha_2 \in L(W_1 \cup W_2)$

$\Rightarrow \alpha_1 + \alpha_2 \in L(W_1 \cup W_2)$

$\therefore \alpha_1 + \alpha_2 \in W_1 + W_2 \Rightarrow \alpha_1 + \alpha_2 \in L(W_1 \cup W_2)$

$\therefore W_1 + W_2 \subset L(W_1 \cup W_2) \longrightarrow ④$

combining eq's ③ & ④, we get

$$L(W_1 \cup W_2) = W_1 + W_2$$

Hence proved

Theorem-11

5M 10M Imp

If S is a subset of a vectorspace V(F) then prove that

(i) S is a subspace of V $\Leftrightarrow$ $L(S) = S$

(ii) $L(L(S)) = L(S)$

proof:-

Given that

V(F) be a vector space and

S be a nonempty subset of V(F)

(i) To prove that S is a subspace of V $\Leftrightarrow$ $L(S) = S$

Suppose S is a subspace of V

By definition of L(S), we have

$$S \subseteq L(S) \longrightarrow ①$$

Let $\alpha \in L(S)$, then

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$, where

$\alpha_1, \alpha_2, \ldots, \alpha_n \in S$ and

$a_1, a_2, \ldots, a_n \in F$

$\because a_1, a_2, \ldots, a_n \in F$ and

$\alpha_1, \alpha_2, \ldots, \alpha_n \in S$ and S is a subspace [S is itself a vector space]

$\Rightarrow a_1\alpha_1, a_2\alpha_2, \cdots a_n\alpha_n \in S$ [by scalar multiplication property in S]

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in S$ [by closure axiom w.r.t + in S]

$\Rightarrow \alpha \in S$

$\therefore \alpha \in L(S) \Rightarrow \alpha \in S$

$\therefore L(S) \subseteq S \rightarrow ②$

combining eq's ① & ②, we have

$L(S) = S$

Converse

Suppose $L(S) = S$

By previous theorem

$L(S)$ is a subspace of $V(F)$

$\therefore S$ is a subspace of $V(F)$

(ii) To prove that $L(L(S)) = L(S)$

we know that

S is a subspace $\Leftrightarrow L(S) = S$

$\because L(S)$ is a subspace of $V(F)$

$\Rightarrow L(L(S)) = L(S)$

Theorem-12

If S, T are subsets of a vector space $V(F)$ then prove that

[~~S subset are equal to~~] 1) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

2) $L(S \cup T) = L(S) + L(T)$

Proof:-

Given that

$V(F)$ be a vector space and

S, T are subsets of $V(F)$

1) To prove that $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

Suppose $S \subseteq T$

Let $\alpha \in L(S)$

$\Rightarrow \alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$, where

$\alpha_1, \alpha_2, \cdots, \alpha_n \in S$, $a_1, a_2, \cdots, a_n \in F$

$\because \alpha_1, \alpha_2, \cdots, \alpha_n \in S \subset T$

$\Rightarrow \alpha_1, \alpha_2, \cdots, \alpha_n \in T$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(T)$

$\Rightarrow \alpha \in L(T)$

$\therefore \alpha \in L(S) \Rightarrow \alpha \in L(T)$

$\therefore L(S) \subseteq L(T)$

2) To prove that $L(S \cup T) = L(S) + L(T)$

Let $\gamma \in L(S \cup T)$

$\Rightarrow \gamma$ can be written as linear combination of elements of $S \cup T$

$\Rightarrow \gamma = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$, where

$\alpha_1, \alpha_2, \ldots, \alpha_n \in S$ & $\beta_1, \beta_2, \ldots, \beta_n \in T$; $a_i, b_i \in F$

$\because \alpha_1, \alpha_2, \ldots, \alpha_n \in S,\ a_1, a_2, \ldots, a_n \in F$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(S)$

lly $b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n \in L(T)$

$\therefore (a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + (b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n) \in L(S) + L(T)$

$\Rightarrow \gamma \in L(S) + L(T)$

$\therefore \gamma \in L(S \cup T) \Rightarrow \gamma \in L(S) + L(T)$

$\therefore L(S \cup T) \subseteq L(S) + L(T) \rightarrow ①$

Let $\delta \in L(S) + L(T)$

$\Rightarrow \delta = x + y$, where $x \in L(S)$, $y \in L(T)$

$\because x \in L(S) \Rightarrow x$ is linear combination of elements of S

$y \in L(T) \Rightarrow y$ is linear combination of elements of T

$\therefore x + y =$ Linear combination of elements of $S \cup T$

$\Rightarrow x + y \in L(S \cup T)$

$\Rightarrow \delta \in L(S \cup T)$

$\therefore \delta \in L(S) + L(T) \Rightarrow \delta \in L(S \cup T)$

$\therefore L(S) + L(T) \subseteq L(S \cup T) \rightarrow ②$

From eq's ① & ②, we have

$L(S \cup T) = L(S) + L(T)$

Imp Linear dependence of vectors ①

Let $V(F)$ be a vector space. A finite subset $\alpha_1, \alpha_2, \ldots, \alpha_n$ of vectors of V is said to be linearly dependent [LD] set of vectors if $\exists$ scalars $a_1, a_2, a_3, \ldots, a_n$, not all zero such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_n\alpha_n = \bar{0}$.

Imp Linear independence of vectors

Let V(F) be a vector space. A finite subset $\alpha_1, \alpha_2, \dots, \alpha_n$ of vectors of V is said to be linearly independent [L I] set if every relation of the form $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0}$ implies $a_1 = 0, a_2 = 0, a_3 = 0, \dots, a_n = 0$

Theorem-13

5M Every superset of a linearly dependent set of vectors is linearly dependent.

proof:-

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a finite set of linearly dependent vectors.

$\Rightarrow \exists$ scalars $a_1, a_2, \dots, a_n$, not all zero, such that

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0} \longrightarrow ①$$

Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n\}$

clearly B is a superset of S.

In view of eq①, consider

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0}$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + 0\cdot\beta_1 + 0\cdot\beta_2 + \dots + 0\cdot\beta_n = \bar{0} \longrightarrow ②$$

In eq②, not all scalars are zero

$\Rightarrow$ The set B is linear dependent

$\therefore$ Every super set of L.D set is L.D

Theorem-14

5M Every nonempty subset of a linearly independent set of vectors is linearly independent.

proof:-

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a finite set of linearly independent set of vectors.

Then every relation

$$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = \bar{0} \longrightarrow ①$$

implies $a_1 = 0, a_2 = 0, \dots, a_n = 0$

Let us consider a set

$$S' = \{\alpha_1, \alpha_2, \dots, \alpha_k\}, \text{ where } 1 \leq k < n$$

consider the equation

$b_1\alpha_1 + b_2\alpha_2 + \cdots + b_K\alpha_K = \bar{0}$

$\Rightarrow b_1\alpha_1 + b_2\alpha_2 + \cdots + b_K\alpha_K + 0\cdot\alpha_{K+1} + \cdots + 0\alpha_n = \bar{0} \to ②$

Since the vectors in eq② are linearly independent

$\Rightarrow$ every scalar involved in eq② is zero

$\Rightarrow b_1 = 0, b_2 = 0, \cdots, b_K = 0$

$\therefore b_1\alpha_1 + b_2\alpha_2 + \cdots + b_K\alpha_K = \bar{0} \Rightarrow b_1 = 0, b_2 = 0, \cdots, b_K = 0$

$\Rightarrow S' = \{\alpha_1, \alpha_2, \cdots, \alpha_K\}$ is L·I

$\therefore$ Every nonempty subset of L·I set is L·I

Theorem-15

A set of vectors which contain zero vector is linearly dependent

proof:-

Let $\alpha_1 = \bar{0}, \alpha_2 \neq \alpha_3 \neq \cdots \neq \alpha_n \neq 0$

and $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$

Now

$1\cdot\alpha_1 + 0\cdot\alpha_2 + 0\cdot\alpha_3 + \cdots + 0\cdot\alpha_n = \bar{0} \Rightarrow$ atleast one scalar $a_1 = 1 \neq 0$

$\therefore S$ is L·D

$\therefore$ Every set containing zero vector is L·D

Theorem-16

A single nonzero vector forms a linearly independent set

proof:- Let $S = \{\alpha\}, \alpha \neq 0$

Then $a\alpha = \bar{0} \Rightarrow a = 0, \forall a \in F$

$\therefore S$ is L·D

$\therefore$ Every singleton set having nonzero vector is L·I

Imp

Theorem-17

Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a subset of the vector space $V(F)$ if $\alpha_i \in S$ is a linear combination of its preceeding vectors then $L(S) = L(S')$

proof:- Given that - $V(F)$ be a vector space and

Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a subset of the vector space $V(F)$

by data,

If $\alpha_i \in S$ is a linear combination of its preceeding vectors i.e $\alpha_i = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} \rightarrow (1)$

Take $S' = \{\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, \alpha_{i+2}, \cdots, \alpha_n\}$

Clearly $S' \subset S$

$\Rightarrow L(S') \subset L(S) \rightarrow (2)$

Let $\beta \in L(S)$

$\Rightarrow \beta$ is L.c of elements of S

$\Rightarrow \beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_i\alpha_i + \cdots + b_n\alpha_n$

$\Rightarrow \beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_i[a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1}] + \cdots + b_n\alpha_n$

$\Rightarrow \beta = [b_1 + b_i a_1]\alpha_1 + [b_2 + b_i a_2]\alpha_2 + \cdots + b_n\alpha_n$

$\Rightarrow \beta =$ L.c of elements of $S' \in L(S')$

$\Rightarrow \beta \in L(S')$

$\therefore \beta \in L(S) \Rightarrow \beta \in L(S')$

$\therefore L(S) \subset L(S') \rightarrow (3)$

From eq's (2) & (3), we have

$L(S) = L(S')$

Hence proved

Note

Let $V(F)$ be a vector space & $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is a finite subset of nonzero vectors of $V(F)$. Then S is L.D $\Leftrightarrow$ some vector $\alpha_k \in S$ can be expressed as a linear combination of its preceeding vectors.

2) If β is a linear combination of a set of vectors $\alpha_1, \alpha_2 \cdots \alpha_n$ then the set $\{\beta, \alpha_1, \alpha_2, \cdots, \alpha_n\}$ is linearly dependent.

5M Imp
11) Show that the system of vectors $(1,3,2), (1,-7,-8), (2,1,-1)$ of $V_3(R)$ is linearly dependent.

Given that

$V_3(R)$ be the vector space and

vectors are $\alpha = (1,3,2)$, $\beta = (1,-7,-8)$, $\gamma = (2,1,-1)$

By writing the coordinates of α, β, γ in rows of a matrix, we get

$$B = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -7 & -8 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 3 & 2 \\ 1 & -7 & -8 \\ 2 & 1 & -1 \end{vmatrix} = 1(7+8) - 3(-1+16) + 2(1+14)$$
$$= 15 - 45 + 30$$
$$= 0$$

$\therefore \det B = 0 \Rightarrow B$ is singular matrix

$\therefore$ The vectors α, β, γ are L.D

12) Show that the system of vectors $(1,2,0), (0,3,1), (-1,0,1)$ of $V_3(Q)$ is linearly independent, where Q is the field of rational numbers

Given that

$V_3(Q)$ be the vector space and

vectors are $\alpha = (1,2,0)$, $\beta = (0,3,1)$, $\gamma = (-1,0,1)$

By writing the coordinates of α, β, γ in rows of a matrix, we get

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 1(3-0) - 2(0+1)$$
$$= 3 - 2$$
$$= 1$$

$\therefore \det B \neq 0 \Rightarrow B$ is non singular matrix

$\therefore$ The vectors α, β, γ are L.I

13) Determine whether the following sets of vectors are linearly independent or linearly dependent

(i) $\{(2,-3), (6,-9)\}$ in $R^2(R)$ (iii) $\{(-1,2,1), (3,0,-1), (-5,4,3)\}$ in $R^3(R)$

(ii) $\{(1,-2,1), (2,1,-1), (7,-4,1)\}$ in $R^3(R)$ (iv) $\{(1,1,0), (1,0,1), (0,1,1)\}$ in $R^3(R)$

(v) $\{(-2,0,3), (1,3,0), (2,4,-1)\}$ in $R^3(R)$

Given that

$R^3(R)$ be the vector space and

vectors are $\alpha = (1,-2,1)$, $\beta = (2,1,-1)$, $\gamma = (7,-4,1)$

By writing the coordinates of α, β, γ in rows of a matrix, we get

$$B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{vmatrix} = 1(1-4)+2(2+7)+1(-8-7)$$
$$= -3+18-15$$
$$= 0$$

$\therefore \det B = 0 \Rightarrow B$ is singular matrix

$\therefore$ The vectors α, β, γ are L.D

(iii)
Given that

$R^3(R)$ be the vector space and

vectors are $\alpha = (-1,2,1)$, $\beta = (3,0,-1)$, $\gamma = (-5,4,3)$

By writing the coordinates of α, β, γ in rows of a matrix, we get

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ -5 & 4 & 3 \end{bmatrix}$$

$$\det B = \begin{vmatrix} -1 & 2 & 1 \\ 3 & 0 & -1 \\ -5 & 4 & 3 \end{vmatrix} = -1(0+4)-2(9-5)+1(12)$$
$$= -4-8+12$$
$$= 0$$

$\therefore \det B = 0 \Rightarrow B$ is singular matrix

$\therefore$ The vectors α, β, γ are L.D

(iv) Given that

$R^3(R)$ be the vector space and

vectors are $\alpha = (1,1,0)$, $\beta = (1,0,1)$, $\gamma = (0,1,1)$

By writing the coordinates of α, β, γ in rows of a matrix, we get

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det B = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(0-1)-1(1-0)+0(1-0)$$
$$= -1-1$$
$$= -2$$

$\therefore \det B = -2 \Rightarrow B$ is non singular matrix

$\therefore$ The vectors α, β, γ are L.I

(v) Given that

$R^3(R)$ be the vector space and

vectors are $\alpha=(-2,0,3)$, $\beta=(1,3,0)$, $\gamma=(2,4,-1)$

By writing the coordinates of α,β,γ in rows of a matrix, we get

$$B=\begin{bmatrix}-2&0&3\\1&3&0\\2&4&-1\end{bmatrix}$$

$$\det B=\begin{vmatrix}-2&0&3\\1&3&0\\2&4&-1\end{vmatrix}=-2(-3)+3(4-6)$$
$$=6-6=0$$

$\therefore \det B=0 \Rightarrow B$ is singular matrix

$\therefore$ The vectors α,β,γ are L.D

14) Determine the set of vectors $\{(2,0,5),(3,-5,8),(4,-2,1),(0,0,1)\}$ is linearly independent or dependent over $R^3(R)$.

Given that: $R^3(R)$ be the vector space and

$S=\{(2,0,5),(3,-5,8),(4,-2,1),(0,0,1)\}\subset R^3(R)$

Let $a,b,c,d\in R$ such that

$a(2,0,5)+b(3,-5,8)+c(4,-2,1)+d(0,0,1)=\bar{0}$

$\Rightarrow (2a,0,5a)+(3b,-5b,8b)+(4c,-2c,c)+(0,0,d)=(0,0,0)$

$\Rightarrow 2a+3b+4c=0 \longrightarrow ①$

$-5b-2c=0 \longrightarrow ②$

$5a+8b+c+d=0 \longrightarrow ③$

eq② $\Rightarrow -5b=2c$

$b=\frac{-2c}{5}$

$eq① \Rightarrow 2a - \frac{6c}{5} + 4c = 0$

$\Rightarrow 10a - 6c + 20c = 0$

$\Rightarrow 10a + 14c = 0$

$\Rightarrow 5a + 7c = 0$

$\Rightarrow 5a = -7c$

$\Rightarrow a = \frac{-7c}{5}$

$eq③ \Rightarrow -7c - \frac{16c}{5} + c + d = 0$

$\Rightarrow -35c - 16c + 5c + 5d = 0$

$\Rightarrow -46c = -5d$

$\Rightarrow d = \frac{46}{5}c$

$\therefore a = \frac{-7c}{5},\ b = \frac{-2c}{5},\ d = \frac{46c}{5}$

$\therefore$ atleast one of a,b,c,d must be not zero

$\therefore$ S is L.D

15) Determine whether the following sets of vectors in $R^4(R)$ is linearly independent (or) dependent

(i) $\{(0,1,0,1), (1,2,3,-1), (1,0,1,0), (0,3,2,0)\}$

(ii) $\{(1,2,-1,1), (0,1,-1,2), (2,1,0,3), (1,1,0,0)\}$

(iii) $\{(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0), (2,1,1,6)\}$

Given that : $R^4(R)$ be the vector space and

vectors are α =

By writing the coordinates of α, β, γ in rows of a matrix, we get

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{bmatrix}$$

$$\det B = \begin{vmatrix} \overset{+}{0} & \overset{-}{1} & \overset{+}{0} & \overset{-}{1} \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 \end{vmatrix} = 0 \begin{vmatrix} \\ \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} + 0 \begin{vmatrix} \\ \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & 2 \end{vmatrix}$$

$= 0 - 1\{-1(2)\} + 0 + \{-3 - 2(2) + 3(3)\}$

$= 2 - 3 - 4 + 9 \quad 2 - 2$

$= \cancel{4 \neq 0} \quad = 0$

$\therefore \det B = 0 \Rightarrow B$ is singular matrix

$\therefore$ The vectors $\alpha, \beta, \gamma, \delta$ are L.D

Given that : $R^4(R)$ be the vector space and

$S = \{(0,1,0,1), (1,2,3,-1), (1,0,1,0), (0,3,2,0)\} \subset R^4(R)$

Let $a,b,c,d \in R$ such that

$a(0,1,0,1) + b(1,2,3,-1) + c(1,0,1,0) + d(0,3,2,0) = \bar{0}$

$\Rightarrow (0,a,0,a) + (b,2b,3b,-b) + (c,0,c,0) + (0,3d,2d,0) = (0,0,0,0)$

$\Rightarrow b+c=0 \rightarrow (1)$

$a+2b+3d=0 \rightarrow (2)$

$3b+c+2d=0 \rightarrow (3)$

$a-b=0 \rightarrow (4)$

From eq (4) , $a=b$

Sub eq $a=b$ in eq (1)

$\Rightarrow a+c=0$

$a=-c$

From eq (3) $-c+2a$.

From eq (1) $b=-c$

From eq (4) $a=-c$

From eq (3) $-3c+c+2d=0$

$-2c=-2d$

$d=c$

From eq (2)

$-c-2c+3d=0$

$-3c=-3d$

$d=c$

$\therefore a=-c,\ b=-c,\ d=c$

$\therefore$ The given vectors are L.D

(ii) Given that : $R^4(R)$ be the vector space and

$S = \{(1,2,-1,1), (0,1,-1,2), (2,1,0,3), (1,1,0,0)\} \subset R^4(R)$

Let $a,b,c,d \in R$ such that

$a(1,2,-1,1) + b(0,1,-1,2) + c(2,1,0,3) + d(1,1,0,0) = \bar{0}$

$\Rightarrow (a,2a,-a,a) + (0,b,-b,2b) + (2c,c,0,3c) + (d,d,0,0) = (0,0,0,0)$

$\Rightarrow a+2c+d=0 \rightarrow (1)$

$2a+b+c+d=0 \rightarrow (2)$

$-a-b=0 \rightarrow (3)$

$a+2b+3c=0 \rightarrow (4)$

From eq (3)

$-a=b$

$a=-b$

From eq ④

$-b+2b+3c=0$

$b+3c=0$

$b=-3c$

$\therefore a=-b$

$a=-(-3c)=3c$

From eq ②

$2(3c)+2(-3c)+c+d=0$

$7c-3c+d=0$

$d=-4c$

$\therefore a=3c,\ b=-3c,\ d=-4c$

$\therefore$ The given vectors are L.D

Given that : $R^4(R)$ be the vector space and

$S=\{(1,1,2,4),(2,-1,-5,2),(1,-1,-4,0),(2,1,1,6)\}$

Let $a,b,c,d \in R$ such that

$a(1,1,2,4)+b(2,-1,-5,2)+c(1,-1,-4,0)+d(2,1,1,6)=\bar{0}$

$\Rightarrow (a,a,2a,4a)+(2b,-b,-5b,2b)+(c,-c,-4c,0)+(2d,d,d,6d)=(0,0,0,0)$

$\Rightarrow a+2b+c+2d=0 \rightarrow ①$

$a-b-c+d=0 \rightarrow ②$

$2a-5b-4c+d=0 \rightarrow ③$

$4a+2b+6d=0 \rightarrow ④$

16) If two vectors are linearly dependent then prove that one of them is a scalar multiple of the other.

Let α, β be two L·D vectors

$\Rightarrow \exists$ two scalars a, b, not all zero, such that

$$a\alpha + b\beta = \bar{0}$$

$\because a, b$ are not both zero [atleast one of them is nonzero]

say $a \neq 0$

Then $a\alpha = -b\beta$

$$\alpha = \left(\frac{-b}{a}\right)\beta$$

$$\Rightarrow \alpha = c\beta, \quad c = -\frac{b}{a}$$

$\Rightarrow \alpha$ is scalar multiple of β by c

$\therefore$ any two L·D must be scalar multiple of other.

17) If α, β, γ are L·I vectors then prove that $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ are also L·I

Let $a, b, c \in R$

consider

$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = \bar{0}$

$\Rightarrow a\alpha + a\beta + b\beta + b\gamma + c\gamma + c\alpha = \bar{0}$

$\Rightarrow (a+c)\alpha + (a+b)\beta + (b+c)\gamma = \bar{0}$

$\because \alpha, \beta, \gamma$ are L·I

$\Rightarrow a+c = 0 \rightarrow ①, \; b+a = 0 \rightarrow ②, \; b+c = 0 \rightarrow ③$

Add ①, eq ② $\Rightarrow a+a+b+c = 0$

$\Rightarrow 2a + 0 = 0 \Rightarrow \boxed{a = 0}$

|ly $b = 0, c = 0$

$\therefore a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = \bar{0} \Rightarrow a = b = c = 0$

$\therefore (\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)$ are L·I where α, β, γ are L·I

18) prove that the set $\{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$ is L·I in $V_4(R)$

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Given that

$S = \{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$

clearly $S \subset V_4(R)$

Take $\alpha = (1,0,0,-1), \beta = (0,1,0,-1), \gamma = (0,0,1,-1), \delta = (0,0,0,1)$

Let $a, b, c, d \in R$ such that

$a\alpha + b\beta + c\gamma + d\delta = \bar{0}$

$\Rightarrow a(1,0,0,-1)+b(0,1,0,-1)+c(0,0,1,-1)+d(0,0,0,1)=\bar{0}$

$\Rightarrow (a,0,0,-a)+(b,b,0,-b)+(0,0,c,-c)+(0,0,0,d)=(0,0,0,0)$

$\Rightarrow a=0 \quad (a,b,c,-a-b-c+d)=(0,0,0,0)$

$\Rightarrow b=0 \quad \Rightarrow a=0,\ b=0,\ c=0,\ -a-b-c+d=0$

$c \quad\quad \Rightarrow d=0$

$\therefore a\alpha+b\beta+c\gamma+d\delta=\bar{0}$ implies $a=0, b=0, c=0, d=0$

$\therefore \alpha,\beta,\gamma,\delta$ are L·I vectors

The set S is L·I set

19) Show that the set $\{1, x, x-x^2\}$ is L·I set of vectors in $F(x)$ over the field of real numbers

Given that

$S=\{1,x,x-x^2\}\subset F(x)$ and R be the field of real numbers

Let $a,b,c\in R$ such that

$a\cdot 1+b\cdot x+c(x-x^2)=0$

$\Rightarrow a+bx+cx-cx^2=0$

$\Rightarrow a+(b+c)x-cx^2=0\cdot 1+0\cdot x+0\cdot x^2$

$\Rightarrow a=0,\ b+c=0,\ -c=0$

$\Rightarrow b=0 \Rightarrow c=0$

$\therefore a\cdot 1+b\cdot x+c(x-x^2)=0 \Rightarrow a=b=c=0$

$\therefore \{1,x,x-x^2\}$ is L·I set

20) Determine the set $\{P_1(x), P_2(x)\}$, where $P_1(x)=1-2x+3x^2$, $P_2(x)=2-3x$ in the vector space of all polynomials over R is L·I or L·D

Given that

$S=\{P_1(x), P_2(x)\}\subset F(x)$ and R be the field of Real numbers

$P_1(x)=1-2x+3x^2,\ P_2(x)=2-3x$

Let $a,b\in R$, such that

$aP_1(x)+bP_2(x)=\bar{0}$

$\Rightarrow a[1-2x+3x^2]+b[2-3x]=0$

$\Rightarrow a-2ax+3ax^2+2b-3bx=0$

$\Rightarrow (a+2b)+(-2a-3b)x+3ax^2=0=0\cdot 1+0\cdot x+0\cdot x^2$

$\Rightarrow a+2b=0 \rightarrow ① \quad -2a-3b=0 \rightarrow ② \quad 3a=0 \rightarrow ③$

$\Rightarrow a=0$

$\Rightarrow b=0$

$\therefore aP_1(x)+bP_2(x)=0 \Rightarrow a=b=0$

$\therefore P_1(x), P_2(x)$ are L·I

$\therefore$ The set S is L·I set

21) under what conditions on the scalar β, the set $\{(\beta,1,0),(1,\beta,1),(0,1,\beta)\} \subset R^3(R)$ is linearly dependent

Given that

$S = \{(\beta,1,0),(1,\beta,1),(0,1,\beta)\} \subset R^3(R)$

By writing the elements of S as rows of a matrix we have

$$B = \begin{bmatrix} \beta & 1 & 0 \\ 1 & \beta & 1 \\ 0 & 1 & \beta \end{bmatrix}$$

S is linearly dependent $\Leftrightarrow$ B is singular

$\Leftrightarrow \det B = 0$

$$\Leftrightarrow \begin{vmatrix} \beta & 1 & 0 \\ 1 & \beta & 1 \\ 0 & 1 & \beta \end{vmatrix} = 0$$

$\Leftrightarrow \beta(\beta^2-1) - 1(\beta) = 0$

$\Leftrightarrow \cancel{\beta}(\beta^2-1-1) = 0 \qquad \beta \neq 0$

$\Leftrightarrow \beta^2 - 2 = 0$

$\Leftrightarrow \beta = \pm\sqrt{2}$

$\therefore$ S is L·D $\Leftrightarrow \beta = \pm\sqrt{2}$

22) ~~B~~ V be the vector space of 2x3 matrices over R then show that the vectors A, B, C forms a L·I set, where $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix}$

Given that

$A = 2$

Given that

V ~~be~~ the vector space of 2x3 matrices over R

Let $a, b, c \in R$, such that

$aA + bB + cC = 0$

$$\Rightarrow a\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + c\begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2a & a & -a \\ 3a & -2a & 4a \end{bmatrix} + \begin{bmatrix} b & b & -3b \\ -2b & 0 & 5b \end{bmatrix} + \begin{bmatrix} 4c & -c & 2c \\ c & -2c & -3c \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a-2c & 4a+5b-3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$2a+b+4c = 0 \rightarrow ①$, $a+b-c=0 \rightarrow ②$, $-a-3b+2c=0 \rightarrow ③$

$3a-2b+c = 0 \rightarrow ④$, $-2a-2c=0 \rightarrow ⑤$, $4a+5b-3c=0 \rightarrow ⑥$

eq ② + ④ $\Rightarrow 4a-b=0 \rightarrow$ ⑦

eq ③ + ⑤ $\Rightarrow -3a-3b=0$

$\Rightarrow a+b=0 \rightarrow$ ⑧

⑦ + ⑧ $\Rightarrow 5a=0 \Rightarrow a=0$

eq ⑧ $\Rightarrow b=0$

eq ⑤ $\Rightarrow c=0$

$\therefore aA+bB+cC=0 \Rightarrow a=b=c=0$

$\therefore \{A, B, C\}$ is L.I

23) Show that the set $\{A, B\}$, where $A=\begin{bmatrix}1 & -2 & 4\\ 3 & 1 & -1\end{bmatrix}$ and $B=\begin{bmatrix}3 & -6 & 12\\ 9 & 3 & -3\end{bmatrix}$ is L.D or L.I over the vector space of 2x3 matrices over R.

Given that

$A=\begin{bmatrix}1 & -2 & 4\\ 3 & 1 & -1\end{bmatrix}$ and $B=\begin{bmatrix}3 & -6 & 12\\ 9 & 3 & -3\end{bmatrix}$

V be the vector space of 2x3 matrices over R

Let $a, b \in R$ such that

$aA+bB=0$

$\Rightarrow a\begin{bmatrix}1 & -2 & 4\\ 3 & 1 & -1\end{bmatrix}+b\begin{bmatrix}3 & -6 & 12\\ 9 & 3 & -3\end{bmatrix}=0$

$\Rightarrow \begin{bmatrix}a & -2a & 4a\\ 3a & a & -a\end{bmatrix}+\begin{bmatrix}3b & -6b & 12b\\ 9b & 3b & -3b\end{bmatrix}=0$

$\Rightarrow \begin{bmatrix}a+3b & -2a-6b & 4a+12b\\ 3a+9b & a+3b & -a-3b\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

$\Rightarrow a+3b=0 \rightarrow$ ①, $-2a-6b=0 \rightarrow$ ②, $4a+12b=0 \rightarrow$ ③

$3a+9b=0 \rightarrow$ ④, $a+3b=0 \rightarrow$ ⑤, $-a-3b=0 \rightarrow$ ⑥

From eq ① $a=-3b$

② $a+3b=0$

$a=-3b$

$\therefore$ Atleast one of a, b are not zero

$\therefore$ S is L.D

24) Express the vector $\alpha=(1,-2,5)$ as a linear combination of the vectors $e_1=(1,1,1)$, $e_2=(1,2,3)$ & $e_3=(2,-1,1)$

Given that

$\alpha=(1,-2,5)$ and

$e_1=(1,1,1)$, $e_2=(1,2,3)$ & $e_3=(2,-1,1)$

Now

$\alpha = ae_1 + be_2 + ce_3 \rightarrow ①$ where $a,b,c \in R$

$\Rightarrow (1,-2,5) = a(1,1,1) + b(1,2,3) + c(2,-1,1)$

$\Rightarrow (1,-2,5) = (a,a,a) + (b,2b,3b) + (2c,-c,c)$

$= (a+b+2c,\ a+2b-c,\ a+3b+c)$

Equating the corresponding terms, we get

$a+b+2c = 1 \rightarrow ②$

$a+2b-c = -2 \rightarrow ③$

$a+3b+c = 5 \rightarrow ④$

eq③ + ④ $\Rightarrow 2a+5b = 3 \rightarrow ⑤$

eq ② + 2 × eq③ $\Rightarrow$ $a+b+\cancel{2c} = 1$

$2a+4b-\cancel{2c} = -4$

$3a+5b = -3 \rightarrow ⑥$

eq⑥ − ⑤ $\Rightarrow \boxed{a=-6}$

eq⑤ $\Rightarrow 5b = 3+12$

$5b = 15 \Rightarrow \boxed{b=3}$

eq ② $\Rightarrow -6+3+2c = 1$

$-3+2c = 1$

$2c = 4$

$\Rightarrow \boxed{c=2}$

Using the values of a,b,c in eq①, we get

$\alpha = -6e_1 + 3e_2 + 2e_3$

i.e α is linear combination of e_1, e_2, e_3

25) Show that the vector $\alpha = (2,-5,3)$ in R^3 cannot be expressed as linear combination of the vectors $e_1 = (1,-3,2)$, $e_2 = (2,-4,-1)$, $e_3 = (1,-5,7)$

Given that

$\alpha = (2,-5,3)$ and

$e_1 = (1,-3,2),\ e_2 = (2,-4,-1),\ e_3 = (1,-5,7)$

Now

$\alpha = ae_1 + be_2 + ce_3 \rightarrow ①$, where $a,b,c \in R$

$\Rightarrow (2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$

$= (a,-3a,2a) + (2b,-4b,-b) + (c,-5c,7c)$

$= (a+2b+c,\ -3a-4b-5c,\ 2a-b+7c)$

Equating the corresponding terms, we get

$a+2b+c=2 \rightarrow (2)$

$3a+4b+5c=5 \rightarrow (3)$

$2a-b+7c=3 \rightarrow (4)$

$eq(2)+2\times(4) \Rightarrow a+\cancel{2b}+c=2$

$$\frac{\begin{array}{r} 4a-\cancel{2b}+14c=6 \end{array}}{5a+15c=8 \rightarrow (5)}$$

$eq(3)-2\times eq(2) \Rightarrow 3a+\cancel{4b}+5c=5$

$$\frac{\begin{array}{r} 2a+\cancel{4b}+2c=4 \end{array}}{a+3c=1 \rightarrow (6)}$$

$eq(5)-5\times eq(6) \Rightarrow 5a+15c=8$

$$\frac{5a+15c=5}{0=3} \Rightarrow \text{which is contradiction}$$

$\therefore \alpha$ cannot be expressed as Linear combination of e_1, e_2, e_3

26) Show that the subset $S=\{(1,0,0),(0,1,0),(0,0,1)\}$ of $V_3(R)$ generate the entire vector space of $V_3(R)$

Given that

$V_3(R)$ be the vector space and

$S=\{(1,0,0),(0,1,0),(0,0,1)\}$

Since $V_3(R)$ is the set of all possible triods over the field R.

$\therefore S \subseteq V_3(R)$

$\Rightarrow L(S) \subseteq V_3(R) \rightarrow (1)$

Let $(a,b,c) \in V_3(R)$

Then

$(a,b,c) = (a,0,0)+(0,b,0)+(0,0,c)$

$= a(1,0,0)+b(0,1,0)+c(0,0,1)$

$=$ L.C of elements of $S \in L(S)$

$\therefore (a,b,c) \in L(S)$

$\therefore (a,b,c) \in V_3(R) \Rightarrow (a,b,c) \in L(S)$

$\Rightarrow V_3(R) \subseteq L(S) \rightarrow (2)$

combining eq's (1) & (2), we get

$V_3(R) = L(S)$

$\therefore$ The vector space $V_3(R)$ is generated by S

27) show that the following set of vectors $S=\{(1,0,0),(1,1,0),(1,1,1)\}$ generates the vector space of $R^3(R)$

Given that

$R^3(R) = V_3(R)$ be the vector space and

$S = \{(1,0,0), (1,1,0), (1,1,1)\}$

Since $V_3(R)$ is the set of all possible triods over the field R

$\therefore S \subseteq V_3(R)$

$\Rightarrow L(S) \subseteq V_3(R) \rightarrow ①$

Let $(a,b,c) \in V_3(R)$

Suppose $x,y,z \in R$ such that

$(a,b,c) = x(1,0,0) + y(1,1,0) + z(1,1,1) \rightarrow ②$

$= (x,0,0) + (y,y,0) + (z,z,z)$

$= (x+y+z,\ y+z,\ z)$

Equating the corresponding terms, we get

$x+y+z = a$ | $y+z = b \Rightarrow y = b-c$ | $z = c$

eq② $\Rightarrow x+b-c+c = a$

$\Rightarrow x = a-b$

put the values of x,y,z in eq②, we get

$(a,b,c) = (a-b)(1,0,0) + (b-c)(1,1,0) + c(1,1,1)$

$=$ L·C of elements of $S \in L(S)$

$\therefore (a,b,c) \in L(S)$

$\therefore (a,b,c) \in V_3(R) \Rightarrow (a,b,c) \in L(S)$

$\therefore V_3(R) \subseteq L(S) \rightarrow ③$

Combining eq① & ②, we get

$R^3(R) = V_3(R) = L(S)$

$\therefore R^3(R)$ is generated by S

28) Show that each of the following sets generates the vector space $R^3(R)$ (i) $S = \{(1,2,3), (0,1,2), (0,0,1)\}$ (ii) $S = \{(1,2,1), (2,1,0), (1,-1,2)\}$

29) In the vector space $R^3(R)$ the set $S = \{(1,0,0), (0,1,0)\}$ then find $L(S)$

Given that

$R^3(R) = V_3(R)$ be the vector space and

$S = \{\underset{\alpha_1}{(1,0,0)}, \underset{\alpha_2}{(0,1,0)}\}$

The Linear span of S is denoted by $L(S)$ & defined as

$L(S) = \{\alpha / \alpha = \sum_{i=1}^{2} a_i\alpha_i\}$

$= \{\alpha = (a,b,c) / \alpha = (a,b,c) = a_1\alpha_1 + a_2\alpha_2\}$

$= \{\alpha = (a, b, c) / \alpha = (a, b, c) = a_1(1,0,0) + a_2(0,1,0)\}$

$= \{\alpha / \alpha = (a, b, c) = (a_1, 0, 0) + (0, a_2, 0)\}$

$= \{\alpha / \alpha = (a_1, a_2, 0)\}$

28) (i) Given that

$R^3(R) = V_3(R)$ be the vector space and

$S = \{(1,2,3), (0,1,2), (0,0,1)\}$

Since $V_3(R)$ is the set of all possible triods over the field R

$\therefore S \subseteq V_3(R)$

$\Rightarrow L(S) \subseteq V_3(R) \longrightarrow ①$

Let $(a, b, c) \in V_3(R)$

suppose $x, y, z \in R$ such that

$(a, b, c) = x(1,2,3) + y(0,1,2) + z(0,0,1) \longrightarrow ②$

$= (x, 2x, 3x) + (0, y, 2y) + (0, 0, z)$

$= (x, 2x+y, 3x+2y+z)$

Equating the corresponding terms, we get

$x = a \mid 2x+y = b \mid, 3x+2y+z = c \mid$

$2a+y = b$

$y = b-2a$

$\Rightarrow 3a + 2(b-2a) + z = c$

$\Rightarrow 3a + 2b - 4a + z = c$

$\Rightarrow -2a + 2b + z = c$

$\Rightarrow z = c + 2a - 2b$

$z = -3a + 2b + c$

put the values of x, y, z in eq ②, we get

$(a, b, c) = a(1,2,3) + (b-2a)(0,1,2) + (c+2a-2b)(0,0,1)$

$=$ L.c of elements of $S \in L(S)$

$\therefore (a, b, c) \in L(S)$

$\therefore (a, b, c) \in V_3(R) \Rightarrow (a, b, c) \in L(S)$

$\therefore V_3(R) \subseteq L(S) \longrightarrow ③$

combining eq ① & ②, we get

$R^3(R) = V_3(R) = L(S)$

$\therefore R^3(R)$ is generated by S

UNIT-II

Basis and Dimensions

Basis of a vector space

Let V(F) be a vector space & S be a non empty subset of V(F) then S is said to be the ~~basis~~ Basis of V(F). If it satisfies following properties

1) S is linearly dependent.
2) V is spanned by S i.e $V(F) = L(S)$ (or) $L(S) = V$

Note
A vector space may have more than one basis

Standard basis

A vector space may have more than one basis & every vector space has a standard basis, they are given by

i) The vector space $R^2(R)$ has a standard basis, given by $\{(1,0), (0,1)\}$

ii) The vector space $R^3(R)$ has a standard basis given by $\{(1,0,0)\ (0,1,0), (0,0,1)\}$.

iii) The vector space $R^4(R)$ has a standard basis given by $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

iv) The general vector space $R^n(R)$ has a standard basis given by $\{e_1, e_2, --- e_n\}$ where $e_1 = (1, 0, --- 0)$, $e_2 = (0, 1, -- 0 --$ e_n $0, 0, ---- 0, 1)$

Finite dimensional vector space

A vector space V(F) is said to be finite dimensional if it has a finite basis (or) A vector space V(F) is said to be finite dimensional if there is a finite subset S in 'V' such that $L(S) = V$

Theorem-1

If V(F) be a finite dimensional vector space then there exists a basis set of V.

proof:-

Given that

Let $V(F)$ be a finite dimensional vector space

Then by def. $\exists$ a finite set

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subset V(F)$ such that

$L(S) = V$

If S is L.I then

S is a basis for $V(F)$

$\left\{ \begin{array}{l} \text{Assume that} \\ \alpha_i \neq 0,\ \forall i = 1, 2, \ldots n \end{array} \right\}$

If S is L.D, Then $\exists$ a vector

$\alpha_i \in S$ such that

α_i can be written as L.c of its preceeding vectors.

Define $S' = S - \{\alpha_i\}$

Then by previous theorem, we have

$L(S) = L(S')$

$\Rightarrow L(S') = V$

If S' is L.I

Then S' is basis for $V(F)$

~~let~~ If S' is L.D, then $\exists$ a vector

$\alpha_j \in S'$ such that

α_j can be written as L.c of its preceeding vectors

Define $S'' = S' - \{\alpha_j\}$

Then $L(S'') = L(S')$

$\Rightarrow L(S'') = V$

If S'' is L.I

Then S'' is basis for $V(F)$

By preceeding in this ~~nu~~ manner after $(n-1)$ steps, we get a singleton set whose vector is non-zero vector such that it is L.I

It forms a basis for $V(F)$

Hence, every finite dimensional vector space there exists a basis set of V

Note:-

$V(F)$ be a finite dimensional vector space & $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a L.I subset of $V(F)$. Then either S is itself a basis

V (or) S can be extended to form a basis of V. (or)
Every linearly independent set is itself a basis or it can be extended to form a basis of the vector space.

Theorem-2

Let $S=\{\alpha_1,\alpha_2,\cdots\alpha_n\}$ be a basis of a finite dimensional vector space V(F). Then for every element $\alpha\in V$, $\exists$ a unique set of scalars $a_1,a_2,\cdots,a_n\in F$ such that $\alpha=a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n$

proof:-

Let V(F) be a finite dimensional vector space and $S=\{\alpha_1,\alpha_2,\cdots\alpha_n\}$ be a basis of V(F)

Let $\alpha\in V$, Then $\exists$ scalars $a_1,a_2,\cdots,a_n\in F$ such that

$\alpha=a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n$

If possible let $\exists$ another set of scalars $b_1,b_2,\cdots,b_n\in F$ such that

$\alpha=b_1\alpha_1+b_2\alpha_2+\cdots+b_n\alpha_n$

Then

$a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n=b_1\alpha_1+b_2\alpha_2+\cdots+b_n\alpha_n$

$\Rightarrow a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n-b_1\alpha_1-b_2\alpha_2-\cdots-b_n\alpha_n=\bar{0}$

$\Rightarrow(a_1-b_1)\alpha_1+(a_2-b_2)\alpha_2+\cdots+(a_n-b_n)\alpha_n=\bar{0}$

$\because$ S is L·I $\Rightarrow \alpha_1,\alpha_2,\cdots,\alpha_n$ are L·I vectors

$\Rightarrow a_1-b_1=0,\ a_2-b_2=0,\cdots,a_n-b_n=0$

$\Rightarrow a_1=b_1,\ a_2=b_2,\cdots,a_n=b_n$

$\therefore \exists$ unique set of scalars such that

$\alpha=a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n$, where $a_1,a_2,\cdots,a_n\in F$

Note:-

Every element of the vector space can be written as L·C of elements of basis

Coordinates

Let $S=\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be the basis set of a finite dimensional vector space V(F). Let $\beta\in V$ be given by $\beta=a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n$, where $a_1,a_2,\ldots,a_n\in F$ then the scalars $(a_1,a_2,\ldots,a_n)$ are called coordinates of the vector β.

Note:-
Coordinates of a vector change with the change of basis.

V.V.Imp 1) Show that the set of vectors $S = \{(1,1,2),(1,2,5),(5,3,4)\}$ do not form a basis of the vector space of $R^3(R)$

Given that

$R^3(R)$ be the vector space and

$S = \{(1,1,2),(1,2,5),(5,3,4)\}$

By writing the elements of S as rows of a matrix, we get

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}_{3\times3}$$

$R_2 \longrightarrow R_2 - R_1$, $R_3 \longrightarrow R_3 - 5R_1$

$$B \cong \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix}$$

$R_3 \longrightarrow \frac{R_3}{(-2)}$

$$B \cong \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$B \cong \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly B is in echelon form [upper triangular matrix form]

one row of B contains only zero's

$\therefore$ rank$(B) < 3$

$\therefore$ The rows of B forms a L.D set

$\therefore$ S is L.D

$\therefore$ S is not a basis for $R^3(R)$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{vmatrix} = 0$$

$\det A = 0$

$\Rightarrow$ Rows of A forms a L.D set

$\therefore$ S is not L.I

S is not a basis of $R^3(R)$

Imp 2) Show that the set of vectors $S = \{(2,1,4),(1,-1,2),(3,1,-2)\}$ forms a basis of $R^3(R)$

Given that

$R^3(R)$ be the vector space and

$S = \{(2,1,4),(1,-1,2),(3,1,-2)\}$

By writing the elements of S as rows of a matrix, we get

$$B=\begin{bmatrix}2 & 1 & 4\\ 1 & -1 & 2\\ 3 & 1 & -2\end{bmatrix}_{3\times 3}$$

$R_1 \leftrightarrow R_2$

$$B\cong\begin{bmatrix}1 & -1 & 2\\ 2 & 1 & 4\\ 3 & 1 & -2\end{bmatrix}$$

$R_2 \to R_2-2R_1,\ R_3\to R_3-3R_1$

$$B\cong\begin{bmatrix}1 & -1 & 2\\ 0 & 3 & 0\\ 0 & 4 & -8\end{bmatrix}$$

$R_2 \to \frac{R_2}{3},\ R_3\to \frac{R_3}{4}$

$$B\cong\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 0\\ 0 & 1 & -2\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$B\cong\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Clearly B is in echelon form

No row of B contains only zero's

$\therefore$ rank$(B)=3$

$\therefore$ The rows of B forms a L.I set

$\therefore$ S is L.I

Clearly, $S\subseteq R^3(R)$

$\Rightarrow L(S)\subseteq R^3(R) \to ①$

Let $\alpha=(a,b,c)\in R^3(R)$

Suppose $\alpha=(a,b,c)=p(2,1,4)+q(1,-1,2)+r(3,1,-2) \to ②$

$=(2p,p,4p)+(q,-q,2q)+(3r,r,-2r)$

$=(2p+q+3r,\ p-q+r,\ 4p+2q-2r)$

Equating the corresponding terms, we get

$2p+q+3r=a \to ③$

$p-q+r=b \to ④$

$4p+2q-2r=c \to ⑤$

Adding ③ & ④

$3p+4r=a+b \to ⑥$

eq⑤ $+ 2\times$ eq④ $\Rightarrow 4p+2q-2r=c$

$$\frac{\begin{aligned}4p+2q-2r&=c\\2p-2q+2r&=2b\end{aligned}}{6p=c+2b}$$

$$p=\frac{c+2b}{6}$$

eq⑥ $\Rightarrow \frac{3(c+2b)}{6}+4r=a+b$

$\Rightarrow c+2b+8r=2a+2b$

$\Rightarrow 8r=2a-c$

$\Rightarrow r=\frac{2a-c}{8}$

eq④ $\Rightarrow \frac{c+2b}{6}-q+\frac{2a-c}{8}=b$

$\Rightarrow 4c+8b-24q+6a-3c=24b$

$\Rightarrow 6a+c-16b=24q$

$\Rightarrow q=\frac{6a-16b+c}{24}$

using p,q,r in eq②, we get

$$\alpha=(a,b,c)=\left(\frac{c+2b}{6}\right)(2,1,4)+\left(\frac{6a-16b+c}{24}\right)(1,-1,2)+\left(\frac{2a-c}{8}\right)(3,1,-2)$$

$=$ L.c of elements of $S\in L(S)$

$\therefore \alpha\in L(S)$

$\therefore \alpha\in R^3(R)\Rightarrow \alpha\in L(S)\Rightarrow R^3(R)\subseteq L(S)\to$ ⑦

combining eq's ① & ⑦, we get : $L(S)=R^3(R)$

Hence S is L.I & $L(S)=R^3(R)$

$\therefore S$ is a basis of $R^3(R)$

Imp 3) show that the set $\{(1,2,1),(2,1,0),(1,-1,2)\}$ forms a basis of $R^3(R)$

Given that

$R^3(R)$ be the vector space and

$S=\{(1,2,1),(2,1,0),(1,-1,2)\}$

By writing the elements of S as rows of a matrix, we get

$$B=\begin{bmatrix}1&2&1\\2&1&0\\1&-1&2\end{bmatrix}$$

$R_2\to R_2-2R_1\quad,\quad R_3\to R_3-R_1$

$$B \cong \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & -3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$B \cong \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

clearly, B is in echelon form [upper triangular matrix form]

No row of B contains only zero's.

$\therefore$ rank(B) = 3

$\therefore$ The rows of B forms a L.I set

$\therefore$ S is L.I

Since $S \subseteq R^3(R)$

$\Rightarrow L(S) \subseteq R^3(R) \to (1)$

Let $\alpha = (a, b, c) \in R^3(R)$

suppose $\alpha = (a, b, c) = p(1,2,1) + q(2,1,0) + r(1,-1,2) \to (2)$

$= (p, 2p, p) + (2q, q, 0) + (r, -r, 2r)$

$= (p+2q+r,\ 2p+q-r,\ p+2r)$

Equating the corresponding terms, we get

$p+2q+r = a \longrightarrow (3)$

$2p+q-r = b \longrightarrow (4)$

$p+2r = c \longrightarrow (5)$

eq (3) $- 2\times$(4) $\Rightarrow$

$$\begin{array}{r} p+2q+r=a \\ 4p+2q-2r=2b \\ - \quad - \quad + \quad - \\ \hline -3p+3r=a-2b \end{array}$$

$\Rightarrow -p+r = \dfrac{a-2b}{3} \to (6)$

Adding eq (5) & (6), we get

$3r = \dfrac{a-2b}{3} + c$

$3r = \dfrac{a-2b+3c}{3}$

$r = \dfrac{a-2b+3c}{9}$

eq (5) $\Rightarrow p + \frac{2}{9}[a-2b+3c] = c$

$9p+2a-4b+6c = 9c \Rightarrow 9p = -2a+4b+3c \Rightarrow p = \frac{1}{9}[-2a+4b+3c]$

eq ④ $\Rightarrow \frac{2}{9}\left[-2a+4b+3c\right]+q-\frac{1}{9}\left[a-2b+3c\right]=b$

$\Rightarrow -4a+8b+6c+9q-a+2b-3c=9b$

$\Rightarrow 9q = 5a-b-3c$

$\Rightarrow q = \frac{1}{9}\left[5a-b-3c\right]$

using the values of p, q, r in eq②, we get

$\alpha = (a,b,c) = \frac{1}{9}\left[-2a+4b+3c\right](1,2,1)+\frac{1}{9}\left[5a-b-3c\right](2,1,0)+$

$\frac{1}{9}\left[a-2b+3c\right](1,-1,2)$

$=$ L.C of elements of $S \in L(S)$

$\therefore \alpha \in L(S)$

$\therefore \alpha \in R^3(R) \Rightarrow \alpha \in L(S)$

$\therefore R^3(R) \subseteq L(S) \rightarrow$ ⑦

From eq's ① & ⑦, we get

$L(S) = R^3(R)$

Hence S is L.I and $L(S)=R^3(R)$

$\therefore S$ is a basis of $R^3(R)$

V.Imp 4) 10m Show that the set $\{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $C^3(C)$. Hence find the coordinates of the vectors $(3+4i, 6i, 3+7i)$

(ii) $(2i, 3+4i, 5)$ (iii) $(6i, 7, 8i)$ (H)

Given vector space is $C^3(C)$ and

$S=\{(1,0,0),(1,1,0),(1,1,1)\}$

By writing the elements of S as rows of a matrix, we get

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$B \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$B \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cong I_3$$

$\therefore$ Rank(B) = 3

$\therefore$ S is L.I

Clearly $S \subseteq C^3(C)$

$\Rightarrow L(S) \subseteq C^3(C) \longrightarrow ①$

Let $\alpha = (a,b,c) = C^3(C)$

suppose $\alpha = (a,b,c) = p(1,0,0) + q(1,1,0) + r(1,1,1) \longrightarrow ②$

$= (p,0,0) + (q,q,0) + (r,r,r)$

$= (p+q+r, q+r, r)$

Equating the corresponding terms, we get

$p+q+r = a \longrightarrow ③$

$q+r = b \longrightarrow ④$

$r = c \longrightarrow ⑤$

eq ④ $q = b-c$

eq ③ $\Rightarrow p+b-c+c = a$

$\Rightarrow p = a-b$

Using the values of p,q,r in eq ②, we get

$\alpha = (a,b,c) = (a-b)(1,0,0) + (b-c)(1,1,0) + c(1,1,1) \longrightarrow ⑥$

$=$ L.C of elements of $S \in L(S)$

$\therefore \alpha \in L(S)$

$\therefore \alpha \in C^3(C) \Rightarrow \alpha \in L(S)$

$\therefore C^3(C) \subseteq L(S) \longrightarrow ⑦$

From eq's ① & ⑦, we get

$L(S) = C^3(C)$

Hence S is L.I and $L(S) = C^3(C)$

$\therefore$ S is a basis for $C^3(C)$

From eq ⑥

The coordinates of (a,b,c) are $(a-b, b-c, c)$

i) coordinates of $(3+4i, 6i, 3+7i) = (3-2i, -3-i, 3+7i)$

ii) coordinates of $(2i, 3+4i, 5) = (-3-2i, -2+4i, 5)$

iii) coordinates of $(6i, 7, 8i) = (6i-7, 7-8i, 8i)$

5) show that the set of vectors $\{(2,1,0),(0,1,2),(-7,2,5)\}$ is not a basis of $R^3(R)$

6) show that the set of vectors $\{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis for $R^3(R)$

Given that

$R^3(R)$ be the vector space and

$S = \{(2,1,0),(0,1,2),(-7,2,5)\}$

By writing the elements of S as rows of a matrix, we get

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -7 & 2 & 5 \end{bmatrix}$$

6) Given that

$R^3(R)$ be the vector space and

$S = \{(1,1,0),(1,0,1)(0,1,1)\}$ and

By writing the elements of S as rows of a matrix, we get

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$B \cong \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$B \cong \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$B \cong \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Clearly B is in echelon form[upper triangular matrix form]

no row of B contains only zero's

$\therefore$ rank$(B) = 3$

$\therefore$ The rows of B forms a L.I set

$\therefore$ S is L.I

$\therefore$ S is ~~not~~ a basis for $R^3(R)$

7) Find the coordinates of $(2,3,4,-1)$ with respect to the basis of $V_4(R)$. $B = \{(1,1,1,2), (1,-1,0,0), (0,0,1,1), (0,1,0,0)\}$

Given Basis of $V_4(R)$ is

$B = \{(1,1,1,2), (1,-1,0,0), (0,0,1,1), (0,1,0,0)\}$
$= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ say

Take $\alpha = (2,3,4,-1)$

and $\alpha_1 = (1,1,1,2)$, $\alpha_2 = (1,-1,0,0)$, $\alpha_3 = (0,0,1,1)$, $\alpha_4 = (0,1,0,0)$

Suppose $a,b,c,d \in R$ such that

$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4$, Then a,b,c,d are called coordinates of α

$\Rightarrow (2,3,4,-1) = a(1,1,1,2) + b(1,-1,0,0) + c(0,0,1,1) + d(0,1,0,0)$
$= (a,a,a,2a) + (b,-b,0,0) + (0,0,c,c) + (0,d,0,0)$
$= (a+b, a-b+d, a+c, 2a+c)$

Equating the corresponding terms, we get

$\Rightarrow a+b=2 \to (1)$ $a-b+d=3 \to (2)$, $a+c=4 \to (3)$, $2a+c=-1 \to (4)$

eq(4) - (3) $\Rightarrow a=-5$

eq(3) $\Rightarrow c=4-a$
$\Rightarrow c=4+5 \Rightarrow c=9$

eq(1) $\Rightarrow b=7$

eq(2) $\Rightarrow -5-7+d=3$
$\Rightarrow d=15$

$\therefore$ The coordinates of α w.r to B are $(a,b,c,d) = (-5,7,9,15)$

8) Find the coordinates of α w.r to the basis set $\{(1,1,1),(-1,1,1),(1,0,-1)\}$, where $\alpha = 4,5,6$

Given Basis of $V_3(R)$ is

$B = \{(1,1,1),(-1,1,1);(1,0,-1)\}$

$= \{\alpha_1, \alpha_2, \alpha_3\}$ say

Take $\alpha = (4,5,6)$ and $\alpha_1 = (1,1,1)$, $\alpha_2 = (-1,1,1)$, $\alpha_3 = (1,0,-1)$

suppose $a,b,c \in R$ such that

$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$, Then a,b,c are called coordinates of α

$(4,5,6) \Rightarrow a(1,1,1) + b(-1,1,1) + c(1,0,-1)$

$\Rightarrow (a,a,a) + (-b,b,b) + (c,0,-c)$

$\Rightarrow (a-b+c,\ a+b,\ a+b-c)$

Equating the equating terms, we get

$a-b+c = 4 \rightarrow ①$, $a+b=5 \rightarrow ②$, $a+b-c=6 \rightarrow ③$

Adding eq ① & ③, we get

$2a = 10 \Rightarrow \boxed{a=5}$

eq ② $5+b=5$

$\boxed{b=0}$

eq ③ $5+0-c=6$

$-c=1$

$\boxed{c=-1}$

$\therefore$ The coordinates of α w.r to B are $(a,b,c) = (5,0,-1)$

9) Find the coordinates of the vector $(1,0,-1)$ w.r to the basis $\{(0,1,-1),(1,1,0),(1,0,2)\}$

Given Basis of $V_3(R)$ is

$B = \{(0,1,-1),(1,1,0),(1,0,2)\}$

$= \{\alpha_1, \alpha_2, \alpha_3\}$ say

Take $\alpha = (1,0,-1)$ and $\alpha_1 = (0,1,-1)$, $\alpha_2 = (1,1,0)$, $\alpha_3 = (1,0,2)$

suppose $a,b,c \in R$ such that

$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$, Then a,b,c are called coordinates of α

$(1,0,-1) \Rightarrow a(0,1,-1) + b(1,1,0) + c(1,0,2)$

$\Rightarrow (0, a, -a) + (b, b, 0) + (c, 0, 2c)$

$\Rightarrow (b+c,\ a+b,\ -a+2c)$

Equating the corresponding terms, we get

$b+c=1 \to ①$, $a+b=0 \to ②$, $-a+2c=-1 \to ③$

From eq② $-$ ① ~~$-b$~~

$a+b-b-c+1=0$

$a-c=-1 \to ④$

Adding ③ & ④

$-\cancel{a}+2c+1+\cancel{a}-c+1=0$

$\boxed{c=-2}$

From eq ① $b-2=1$ $\boxed{b=3}$ From ② $\boxed{a=-3}$

10) Find the coordinates of $\alpha=(1,2,3)$ w.r to the basis $\{(1,1,1), (-1,1,0), (1,0,-1)\}$

Given Basis of $V_3(R)$ is

$B = \{(1,1,1), (-1,1,0), (1,0,-1)\}$

$= \{\alpha_1, \alpha_2, \alpha_3\}$ say

Take $\alpha=(1,2,3)$ and $\alpha_1=(1,1,1)$, $\alpha_2=(-1,1,0)$, $\alpha_3=(1,0,-1)$

suppose $a,b,c \in R$ such that

$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$, Then a,b,c are called coordinates of α

$(1,2,3) \Rightarrow a(1,1,1) + b(-1,1,0) + c(1,0,-1)$

$\Rightarrow (a,a,a) + (-b,b,0) + (c,0,-c)$

$\Rightarrow (a-b+c,\ a+b,\ a-c)$

Equating the corresponding terms, we get

$a-b+c=1 \to ①$, $a+b=2 \to ②$, $a-c=3 \to ③$

eq② $-$ ③

$a+b-2-a+c+3=0$

$b+c=-1 \to ④$

eq① + ②

$a-\cancel{b}+c-1+a+\cancel{b}-2=0$

$2a+c=3 \to ⑤$

$c=3-2a \to ⑤$

⑤ in ③

$a-3+2a=3$

$3a=6 \Rightarrow \boxed{a=2}$

From ②

$2+b=2$

$\boxed{b=0}$

From ①

$2-0+c=1$

$\boxed{c=-1}$

11) under what conditions on the scalar $\alpha \in R$ the set $\{(0,1,\alpha), (\alpha,0,1), (\alpha,1,1+\alpha)\}$ is a basis of $R^3(R)$

Given set is

$S = \{(0,1,\alpha), (\alpha,0,1), (\alpha,1,1+\alpha)\}$

By writing the elements of S as rows of a matrix, we get

$$B = \begin{bmatrix} 0 & 1 & \alpha \\ \alpha & 0 & 1 \\ \alpha & 1 & 1+\alpha \end{bmatrix}$$

we know that Basis is a L.I set

$\Rightarrow \det B \neq 0$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \alpha \\ \alpha & 0 & 1 \\ \alpha & 1 & 1+\alpha \end{vmatrix} \neq 0$$

$\Rightarrow -1(\cancel{\alpha}+\alpha^2-\cancel{\alpha})+\alpha(\alpha) \neq 0$

$\Rightarrow -\alpha^2+\alpha^2 \neq 0$

$\Rightarrow 0 \neq 0$

This is a contradiction

For any value of α, S is not L.I set

$\therefore$ S is not a basis for any α

5M VV Imp 10M Theorem-3 [In variance Theorem]

Let V(F) be a finite dimensional vector space. Then any two basis of V have same number of elements.

proof:-

Given that:

V(F) be a finite dimensional vector space

Let $A = \{\alpha_1, \alpha_2, ---, \alpha_n\}$ and

$B = \{\beta_1, \beta_2, ---, \beta_m\}$ are two bases of V(F)

Here no. of elements in A = n

and no. of elements in B = m

we have to prove that

$n = m$

Since A is a basis of V(F)

Then [by def of basis]

A is L.I and $L(A) = V(F)$

Since B is a basis of $V(F)$

$\Rightarrow$ B is L·I and $L(B) = V(F)$

Since B is L·I and $L(A) = V(F)$ A is a basis of $V(F)$

$\Rightarrow m \leq n \longrightarrow (1)$

Since A is L·I and $L(B) = V(F)$ B is a basis of $V(F)$

$\Rightarrow n \leq m \longrightarrow (2)$

combining eq's (1) & (2), we get

$m = n$

$\Rightarrow n(A) = n(B)$

$\therefore$ Any two basis of $V(F)$ contains same no. of elements

Note:-

The no. of elements in a L·I set is less than or equal to the no. of elements in the basis

Dimension of vector space

Let $V(F)$ be a finite dimensional vector space. The no. of elements in any basis of V is called the dimension of V & is denoted by $\dim(V)$.

i.e $\dim(V)$ = The no. of elements in any basis of $V(F)$

Ex-1

We know that, the set $\{(1,0),(0,1)\}$ is a basis for $V_2(R) = R^2(F)$

$\therefore$ The $\dim(V_2(R)) = 2$

2) W.K.T, the set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is basis for $V_3(R) = R^3(R)$

$\therefore$ The $\dim(V_3(R)) = 3$

Theorem-4

- Every set of $n+1$ or more vectors in an n dimensional vector space is L·D

Proof:-

Let $V(F)$ be a vector space having dimension 'n'

i.e $\dim(V) = n$

$\Rightarrow$ every basis of $V(F)$ contains n elements only

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}\}$ be a subset of $V(F)$ having $(n+1)$ elements

If S is L·I $\Rightarrow$ S is itself a basis for $V(F)$ (or) it can be extended to form a basis for V(F)

$\Rightarrow$ In either case $\dim(V) \geq n+1$

This is a contradiction to the fact $\dim(V)=n$

$\therefore$ S is L.D

Hence, every subset of $(n+1)$ or more elements of an n dimensional vector space is L.D

Theorem-5

5M Let V(F) be a finite dimensional vector space of dimension 'n'. Then any set of n L.I vectors in V forms a basis of V.

proof:-

Let V(F) be a vector space having dimension 'n'

i.e $\dim(V)=n$

$\Rightarrow$ every basis of V(F) contains n elements only

Let $S=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a L.I subset of V(F) having 'n' elements.

If S is a basis for V(F)

Then there is nothing to prove

If S is not a basis & S is L.I

$\Rightarrow$ S can be extended to form a basis of V(F)

$\Rightarrow \dim(V) > n$

$\Rightarrow$ This is a contradiction to the fact $\dim(V)=n$

Clearly, S is a basis for V(F)

Theorem-6

Let V(F) be a finite dimensional vector space of dimension n. Let S be the subset of n vectors of V such that $L(S)=V$. Then S is a basis of V(F).

proof:-

Let V(F) be a vector space having dimension 'n'

i.e $\dim(V)=n$

$\Rightarrow$ every basis of V(F) contains n elements only

Let $S=\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a subset of V(F) having 'n' elements

Such that $L(S)=V$

we have to P.T S is a basis for V(F)

If S is L·I $\Rightarrow$ S is a basis for $V(F)$

If S is L·D $\Rightarrow$ Then any proper subset of S forms a L·I set, which is a basis of $V(F)$

$\Rightarrow \dim(V) < n$

This is a contradiction [The-1]

$\therefore$ S is L·I

Then S is a basis for $V(F)$

Note

1) If V is a finite dimensional vector space of dimension n then V cannot be generated by a set of vectors whose no·of elements is less than n.

2) V is a vector space which is spanned by a finite set of vectors $\{\beta_1, \beta_2, \ldots, \beta_n\}$ then any independent set of vectors in V is finite & contains not more than n elements.

Dimension of a subspace

Let $V(F)$ be a finite dimensional vector space of dimension n & W be a subspace of V. Then the dimension of W is defined as

dimension of $W \leq \dim(V)$

i·e $\dim(W) \leq n$

Theorem-7

$V(F)$ be a finite dimensional vector space of dimension n & W be a subspace of V. Then W is a finite dimensional vector space such that $\dim(W) \leq n$

proof:-

Given that

$V(F)$ be the finite dimensional vector space of dimension n.

Given that

W is a subspace of $V(F)$

$\therefore$ W is itself a vector space over F

Since $\dim(V) = n$

$\Rightarrow$ every basis of $V(F)$ has 'n' elements only

Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of $V(F)$

$\because$ B is basis of V(F)

$\Rightarrow$ B is L·I and $L(B) = V$

Since $L(B) = V$

$\Rightarrow$ every element of V can be written as L·C of elements of

Since $W \subset V \Rightarrow$ every element of W is an element of V

$\Rightarrow$ every element of W can be written as L·C of elements of B

$\Rightarrow L(B) = W$ & B is L·I

$\Rightarrow$ B is Basis of W(F) (or) any subset of B forms a basis of W

$\therefore$ In either case, the basis of W contains $\leq n$ elements

$\therefore \dim(W) \leq n$

✓ Theorem-8

VIMP 10M Let W_1 & W_2 be two subspaces of a finite dimensional vector space V(F). Then $\dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

proof:-

Let V(F) be the finite dimensional vector space and W_1, W_2 are any two subspaces of V(F)

$\because W_1, W_2$ are two subspaces of V(F)

$\Rightarrow W_1+W_2$ and $W_1 \cap W_2$ are also subspaces of V(F)

$\because$ V(F) is finite dimensional

$\Rightarrow W_1+W_2$ and $W_1 \cap W_2$ are also finite dimensional

ie They have a finite basis

Let $S = \{\alpha_1, \alpha_2, ---, \alpha_n\}$ be a basis of $W_1 \cap W_2$

$\Rightarrow$ S is L·I and $L(S) = W_1 \cap W_2$ and $\dim(W_1 \cap W_2) = n$

Since $W_1 \cap W_2 \subset W_1$ and $W_1 \cap W_2 \subset W_2$

$\Rightarrow$ S is L·I subset of W_1

$\Rightarrow$ S can be extended to form a basis of W_1

Let $S' = \{\alpha_1, \alpha_2, ---, \alpha_n, \beta_1, \beta_2, ---, \beta_m\}$ be an extension of S & a basis of W_1

$\Rightarrow S'$ is L·I & $\dim(W_1) = n+m$

$\because W_1 \cap W_2 \subset W_2$

$\Rightarrow$ S is L·I subset of W_2

$\Rightarrow S$ can be extended to form a basis of W_2

Let $S'' = \{\alpha_1, \alpha_2, ---, \alpha_n, \gamma_1, \gamma_2, --, \gamma_k\}$ be an extension of S and a basis of W_2

$\Rightarrow S''$ is L·I and $\dim(W_2) = n+k$

Now $\dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = n+m+n+k-n$

$= n+m+k$

we have to prove that

$\dim(W_1+W_2) = n+m+k$

consider a set

$B = \{\alpha_1, \alpha_2, ---, \alpha_n, \beta_1, \beta_2, ---, \beta_m, \gamma_1, \gamma_2, --, \gamma_k\}$

If we P·T B is basis for W_1+W_2

Then

$\dim(W_1+W_2) = n+m+k$

To prove that B is L·I

consider an equation

$a_1\alpha_1 + a_2\alpha_2 + --- + a_n\alpha_n + b_1\beta_1 + b_2\beta_2 + -- + b_m\beta_m + c_1\gamma_1 + c_2\gamma_2 + --- + c_k\gamma_k = 0 \longrightarrow ①$

where a_i's, b_i's, c_i's $\in F$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + -- + b_m\beta_m = -a_1\alpha_1 - a_2\alpha_2 - --- - a_n\alpha_n - c_1\gamma_1 - c_2\gamma_2 - --- - c_k\gamma_k$

$=$ L·c of elements of $S'' \in L(S'') = W_2$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + -- + b_m\beta_m \in W_2 \longrightarrow ②$

Also

and $b_1\beta_1 + b_2\beta_2 + -- + b_m\beta_m = b_1\beta_1 + b_2\beta_2 + -- + b_m\beta_m + 0\cdot\alpha_1 + 0\cdot\alpha_2 + -- + 0\alpha_n$

$=$ L·c of elements of $S' \in L(S') = W_1$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + -- + b_m\beta_m \in W_1 \longrightarrow ③$

combining eq ② & ③, we get

$\Rightarrow b_1\beta_1 + b_2\beta_2 + --- + b_m\beta_m \in W_1 \cap W_2$

Since $b_1\beta_1 + b_2\beta_2 + --- + b_m\beta_m \in W_1 \cap W_2$ & S is a basis for $W_1 \cap W_2$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + --- + b_m\beta_m = d_1\alpha_1 + d_2\alpha_2 + --- + d_n\alpha_n$, d_i's $\in F$

$\Rightarrow b_1\beta_1 + b_2\beta_2 + -- + b_m\beta_m - d_1\alpha_1 + d_2\alpha_2 - --- - d_n\alpha_n = \bar{0}$ & S' is L·I

$\Rightarrow b_1 = 0, b_2 = 0, ---, b_m = 0, d_1 = 0, d_2 = 0, ---, d_n = 0$

Then eq ① $\Rightarrow$

$a_1\alpha_1 + a_2\alpha_2 + --- + a_n\alpha_n + c_1\gamma_1 + c_2\gamma_2 + -- + c_k\gamma_k = 0$ & S'' is L·I

$\Rightarrow a_1=0, a_2=0, \ldots, a_n=0, c_1=0, c_2=0, \ldots, c_K=0$

$\therefore a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n+b_1\beta_1+b_2\beta_2+\cdots+b_m\beta_m+c_1\gamma_1+c_2\gamma_2+\cdots+c_K\gamma_K$

implies

$a_1=a_2=\cdots=a_n=b_1=b_2=\cdots=b_m=c_1=c_2=\cdots=c_K=0$

$\therefore B=\{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta_1,\beta_2,\ldots,\beta_m,\gamma_1,\gamma_2,\ldots,\gamma_K\}$ is L-I

$\therefore B$ is L-I set.

(ii) To prove that $L(B)=W_1+W_2$

We know that:

$W_1 \subset W_1+W_2$ and $W_2 \subset W_1+W_2$

$\Rightarrow B \subset W_1+W_2$ $(S' \subset W, S'' \subset W_2)$

$\Rightarrow L(B) \subseteq W_1+W_2 \longrightarrow (4)$

Let $\gamma \in W_1+W_2$

$\Rightarrow \gamma=\alpha+\beta$, where $\alpha \in W_1$, $\beta \in W_2$

$\because \alpha \in W_1$ & S' is a basis for W_1

$\Rightarrow \alpha$ can be written as L.C of elements of S'

$\because \beta \in W_2$ & S'' is a basis for W_2

$\Rightarrow \beta$ can be written as L.C of elements of S''

$\Rightarrow \gamma=\alpha+\beta=$ L.C of elements of S' & S'' $(\subset B)$

$\Rightarrow \gamma$ is L.C of elements of $B \in L(B)$

$\therefore \gamma \in L(B)$

$\therefore \gamma \in W_1+W_2 \Rightarrow \gamma \in L(B)$

$\therefore W_1+W_2 \subseteq L(B) \longrightarrow (5)$

combining eq's (4) & (5), we get

$$L(B)=W_1+W_2$$

From case (1), case (2):

B is L-I and $L(B)=W_1+W_2$

$\Rightarrow B$ is a basis for W_1+W_2

Now $\dim(W_1+W_2)$ = no. of elements in the basis B of W_1+W_2

$= n+m+K$

$= \dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2)$

$\therefore \dim(W_1+W_2)=\dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2)$

Coset

Let $V(F)$ be a vector space and W be a subspace of $V(F)$.
Then for any element $\alpha \in V$, the set
$W+\alpha = \{x+\alpha / x \in W\}$ is called the right coset of W in V generated by α.
The set $\alpha+W = \{\alpha+x / x \in W\}$ is called the left coset of W in V generated by α.

Note

Since $(W,+)$ is a subgroup of the abelian group $(V,+)$.
Then $x+\alpha = \alpha+x$
$\therefore$ $W+\alpha$ is simply called coset of W in V generated by α

2) For $\bar{0} \in V$, $\bar{0}+W = W$
i.e. W is itself a coset generated by $\bar{0}$

3) If $\alpha \in W$ then $\alpha+W = W+\alpha = W$.

4) If $\alpha+W$ & $\beta+W$ are two cosets of W in V then
$\alpha+W = \beta+W \Leftrightarrow \alpha-\beta \in W$

5) Any two cosets of W in V are either identical or disjoint

Quotient set

Let $V(F)$ be a vector space and W be a subspace of V.
Then the set of all cosets of W in V is denoted by $\frac{V}{W} = \{W+\alpha / \alpha \in V\}$, is called quotient set of V by W.

Theorem-9

Imp 10M

Let W be a subspace of a finite dimensional vector space $V(F)$ then $\dim\left(\frac{V}{W}\right) = \dim(V) - \dim(W)$

Proof:-

Given that

$V(F)$ is a finite dimensional vector space and
W is a subspace of $V(F)$
Since $V(F)$ is finite dimensional
$\Rightarrow$ W is also finite dimensional
$\Rightarrow$ W has a finite basis, say

$B = \{\beta_1, \beta_2, ---, \beta_n\}$

i.e B is a basis for W

$\Rightarrow B$ is L.I and $L(B) = W$ and

$\therefore dim(W) = $ no.of elements in $B = n$

$\because B$ is L.I subset of $W \subset V$

$\Rightarrow B$ can be extended to form a basis of $V(F)$

$\therefore S = \{\beta_1, \beta_2, ---, \beta_n, \gamma_1, \gamma_2, ---, \gamma_m\}$ be an extension of B and a basis of $V(F)$

$\because S$ is basis of $V(F)$

$\Rightarrow S$ is L.I & $L(S) = V$ and

$dim(V) =$ no.of elements in the basis S of $V(F)$

$= n+m$

Now

$dim(V) - dim(W) = n+m-n = m$

$\therefore dim(V) - dim(W) = m.$

We know that

$\frac{V}{W}$ is the quotient set which is the set of all cosets of W in V

i.e $\frac{V}{W} = \{x+W / x \in V\}$ and

$\frac{V}{W}$ is a vector space w.r. to coset addition & multiplication defined as follows.

1) $(\alpha+W) + (\beta+W) = (\alpha+\beta)+W$

2) $a(\alpha+W) = a\alpha+W, \ \forall \ \alpha+W, \beta+W \in \frac{V}{W}$ and

we have to prove that

$$dim\left(\frac{V}{W}\right) = m$$

consider a set $S' = \{\gamma_1+W, \gamma_2+W, ---, \gamma_m+W\}$

clearly $S' \subset \frac{V}{W}$

If we P.T S' is a basis for $\frac{V}{W}$, Then

$$dim\left(\frac{V}{W}\right) = m$$

(i) To prove that S' is L.I

Let us consider an equation

$a_1(\gamma_1+W) + a_2(\gamma_2+W) + \cdots + a_m(\gamma_m+W) = \bar{0}+W \rightarrow ②$ where $a_1, a_2, \ldots, a_m \in F$

$\Rightarrow (a_1\gamma_1+W) + (a_2\gamma_2+W) + \cdots + (a_m\gamma_m+W) = \bar{0}+W$

$\Rightarrow (a_1\gamma_1 + a_2\gamma_2 + \cdots + a_m\gamma_m) + W = W$

$\Rightarrow a_1\gamma_1 + a_2\gamma_2 + \cdots + a_m\gamma_m \in W$ & B is a basis of W

$\Rightarrow a_1\gamma_1 + a_2\gamma_2 + \cdots + a_m\gamma_m = b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n$

$\Rightarrow a_1\gamma_1 + a_2\gamma_2 + \cdots + a_m\gamma_m - b_1\beta_1 - b_2\beta_2 - \cdots - b_n\beta_n = \bar{0}$ & S is L.I

$\Rightarrow a_1=0,\ a_2=0, \ldots a_m=0,\ b_1=0,\ b_2=0, \ldots b_n=0$

$\therefore a_1(\gamma_1+W) + a_2(\gamma_2+W) + \cdots + a_m(\gamma_m+W) = \bar{0}+W \Rightarrow a_1=0, a_2=0, \ldots a_m=0$

$\therefore \{\gamma_1+W, \gamma_2+W, \ldots, \gamma_m+W\}$ is L.I

$\therefore S'$ is L.I

(ii) To prove that $L(S') = \frac{V}{W}$

Since, $S' = \{\gamma_1+W, \gamma_2+W, \ldots, \gamma_m+W\}$

clearly $S' \subset \frac{V}{W}$

$\Rightarrow L(S') \subset \frac{V}{W} \rightarrow ③$

Let $\gamma+W \in \frac{V}{W}$, where $\gamma \in V$ and

Since $S = \{\beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots \gamma_m\}$ is a basis of V(F)

$\Rightarrow \exists$ scalars c_i's, d_i's such that

$\gamma = c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n + d_1\gamma_1 + d_2\gamma_2 + \cdots + d_m\gamma_m \rightarrow ④$

Now

$\gamma+W = (c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n) + (d_1\gamma_1 + d_2\gamma_2 + \cdots + d_m\gamma_m) + W$

$= (c_1\beta_1+W) + (c_2\beta_2+W) + \cdots + (c_n\beta_n+W) + (d_1\gamma_1+W) + (d_2\gamma_2+W) + \cdots + (d_m\gamma_m+W)$

$= c_1(\beta_1+W) + c_2(\beta_2+W) + \cdots + c_n(\beta_n+W) + d_1(\gamma_1+W) + d_2(\gamma_2+W) + \cdots + d_m(\gamma_m+W) \rightarrow ⑤$

$\because \{\beta_1, \beta_2, \ldots \beta_n\}$ is a basis for W

$\Rightarrow \beta_1, \beta_2, \ldots, \beta_n \in W$

$\Rightarrow \beta_1+W=W,\ \beta_2+W=W, \ldots \beta_n+W=W$

$\alpha+W = \beta+W \Leftrightarrow \alpha-\beta \in W$

put $\beta=0$

$\alpha+W = W \Leftrightarrow \alpha \in W$

$\bar{0}+W$

Then eq ⑤ $\Rightarrow$

$\gamma+W = W + d_1(\gamma_1+W) + d_2(\gamma_2+W) + \cdots + d_m(\gamma_m+W)$

$\therefore \bar{0}+W = W$ is zero element in $\frac{V}{W}$.

$\gamma+W = d_1(\gamma_1+W) + d_2(\gamma_2+W) + \cdots + d_m(\gamma_m+W)$

$=$ L.c of elements of $S' \in L(S')$

$\therefore \gamma+W \in L(S')$

$\because \gamma+W \in \frac{V}{W} \Rightarrow \gamma+W \in L(S')$

$\Rightarrow \frac{V}{W} \subset L(S') \rightarrow ⑥$

From eq ③ & ⑥, we have

$\therefore \frac{V}{W} = L(S')$

$\therefore$ From case ①, ② : S' is L.I & $L(S') = \frac{V}{W}$

$\therefore S'$ is a basis of $\frac{V}{W}$

Then $\dim\left(\frac{V}{W}\right) =$ no. of elements in the basis of S' of $\frac{V}{W}$

$= m \rightarrow ⑦$

From eq's ① & ⑦ we have

$\dim\left(\frac{V}{W}\right) = \dim(V) - \dim(W)$

12) If W is a subspace of $V_4(R)$ generated by the vectors $\{(1,-2,5,-3)\,(2,3,1,-4)\,(3,8,-3,-5)\}$ then find a basis of W & its dimension

Given that

$V_4(R)$ be the vector space and

W is a subspace of $V_4(R)$ & W is generated by a set of vectors.

$S = \{(1,-2,5,-3)\,(2,3,1,-4)\,(3,8,-3,-5)\}$

i.e; $L(S) = W$

By writing the elements of S as rows of a matrix, we have

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$

$$A \cong \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{2} - R_2$$

$$A \cong \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A is in echelon form

$\therefore$ The non-zero rows of A forms a L.I set

Then $s' = \{ (1,-2,5,-3)\ (0,7,-9,2) \}$ forms a basis of W

$\therefore \dim(W) = 2$

13) If W is a subspace of $R^4(R)$ generated by the vectors $(1,1,0,-1)\ (1,2,3,0)$ & $(2,3,3,-1)$ then find the basis of W & its dimension

Given that

$R^4(R) = V_4(R)$ is the vector space and

W be the subspace of $R^4(R)$, generated by a set of vectors

$S = \{ (1,1,0,-1)\ (1,2,3,0)\ (2,3,3,-1) \}$

By writing the elements of S as rows of a matrix we have

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1 \; ; \; R_3 \rightarrow R_3 - 2R_1$$

$$A \cong \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$A \cong \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, A is in echelon form [upper triangular matrix form]

∴ The non-zero rows of A forms a L·I set

Then $S' = \{(1,1,0,-1)\ (0,1,3,1)\}$ forms a basis of W

S' is a L·I subset of W

so it forms a basis of W

∴ $\dim(W)$ = no. of elements in $S' = 2$

14) If W is a subspace of $V_4(R)$ generated by the vectors $(1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)$ then find basis of W and dim of W

Given that

$V_4(R)$ be the vector space and

W is a subspace of $V_4(R)$ & W is generated by a set of vectors

$S = \{(1,2,2,-2)\ (2,3,2,-3)\ (1,3,4,-3)\}$

i.e., $L(S) = W$

By writing the elements of S as rows of a matrix, we have

$$A = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

$R_3 \to R_3 - R_1$, $R_2 \to R_2 - 2R_1$

$$\cancel{A \cong \begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ -1 & 0 & 2 & 0 \end{bmatrix}} \qquad A \cong \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A \cong \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

clearly, A is in echelon form [upper triangular matrix form]

∴ The non-zero rows of A forms a L·I set

Then $S' = \{(1,1,0,-1)\ (0,1,3,1)\}$ is a LI subset of W

so it forms a basis of W

∴ $\dim(W)$ = no. of elements of $S' = 2$

15) Let W_1 & W_2 be two subspaces of R^4 such that $W_1 = \{(a,b,c,d) / b-2c+d=0\}$ & $W_2 = \{(a,b,c,d) / a=d, b=2c\}$ then find the basis & dimension of W_1, W_2, $W_1 \cap W_2$ & W_1+W_2

VV Imp

Given that

$R^4(R) = V_4(R)$ be the vector space and

W_1 & W_2 be the subspaces of R^4 such that

$W_1 = \{(a,b,c,d) / b-2c+d=0\}$ and

$W_2 = \{(a,b,c,d) / a=d, b=2c\}$

(i) Let $\alpha = (a,b,c,d) \in W_1$, then

$\Rightarrow b-2c+d=0$

$\Rightarrow b = 2c-d$

Now

$\alpha = (a,b,c,d) = (a, 2c-d, c, d)$

$= (a,0,0,0) + (0,2c,c,0) + (0,-d,0,d)$

$= a(1,0,0,0) + c(0,2,1,0) + d(0,-1,0,1)$

$=$ L.C of elements of S

$= \{(1,0,0,0)\ (0,2,1,0)\ (0,-1,0,1)\}$

$\therefore$ The set $S = \{(1,0,0,0), (0,2,1,0)\ (0,-1,0,1)\}$ spans W_1

[i.e $L(S) = W_1$]

By writing the vectors of S as rows of a matrix, we get

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$

$$A \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Clearly, A is in echelon form

$\therefore$ The non-zero rows of A forms a L.I set & a basis for W_1

$\therefore \dim W_1 =$ no. of non zero rows in $A = 3$

and basis of $W_1 = \{(1,0,0,0), (0,2,1,0), (0,0,1,2)\}$

(ii) Given

$W_2 = \{(a,b,c,d) / a = d, b = 2c\}$

Let $\beta = (a,b,c,d) \in W_2$, then

$\therefore a = d, b = 2c$, then

Now,

$$\beta = (a,b,c,d) = (d,2c,c,d)$$
$$= (d,0,0,d) + (0,2c,c,0)$$
$$= d(1,0,0,1) + c(0,2,1,0)$$
$$= \text{L.c of elements of } S'$$
$$S' = \{(1,0,0,1), (0,2,1,0)\}$$

$\therefore$ The set $S' = \{(1,0,0,1), (0,2,1,0)\}$ spans W_2 [i.e; $L(S') = W_2$]

$\therefore \dim(W_2) = $ no. of elements of $S' = 2$

& basis of $W_2 = \{(1,0,0,1), (0,2,1,0)\}$

(iii) Now

$W_1 \cap W_2 = \{(a,b,c,d) / b - 2c + d = 0, a = d, b = 2c\}$

Let us take $\gamma = (a,b,c,d) \in W_1 \cap W_2$

$$\Rightarrow b - 2c + d = 0, a = d, b = 2c$$
$$\Rightarrow b = 2c - d \Rightarrow 2c = 2c - d$$
$$\Rightarrow d = 0 \Rightarrow a = 0$$

$$\gamma = (a,b,c,d) = (0,2c,c,0)$$
$$= c(0,2,1,0)$$
$$= \text{L.c of one vector } (0,2,1,0)$$

$\therefore \dim(W_1 \cap W_2) = 1$ & basis of $W_1 \cap W_2 = \{(0,2,1,0)\}$

Now

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$
$$= 3 + 2 - 1$$
$$= 4$$

$\therefore \dim(W_1 + W_2) = 4$

16) Let W_1, W_2 be any two subspaces of a vector space $R^4(R)$ generated by $\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$ respectively. Then find the dimensions of W_1, W_2 & W_1+W_2 & $W_1 \cap W_2$

17) If P and Q are two subspaces of $R^4(R)$ defined by $P=\{(a,b,c,d) / b+c+d=0\}$ & $Q=\{(a,b,c,d)/ a+b=0, c=2d\}$ then find the dimension & basis of P, Q & $P\cap Q$

Given $R^4(R)$ is a vector space and P and Q are two subspaces of $R^4(R)$ defined by

$$P=\{(a,b,c,d) / b+c+d=0\}$$

and $Q=\{(a,b,c,d) / a+b=0, c=2d\}$

Let $\alpha=(a,b,c,d)\in P$

$\Rightarrow b+c+d=0$

$\Rightarrow b=-c-d$

Now

$\alpha=(a,b,c,d)=(a,-c-d,c,d)$

$=(a,0,0,0)+(0,-c,c,0)+(0,-d,0,d)$

$=a(1,0,0,0)+c(0,-1,1,0)+d(0,-1,0,1)$

$=$ L·c of elements of $S=\{(1,0,0,0),(0,-1,1,0),(0,-1,0,1)\}$

$\therefore \dim P=3$

Now,

Let $\beta=(a,b,c,d)\in Q$

$\Rightarrow a+b=0, c=2d$

$\Rightarrow a=-b$

Now, $\beta=(-b,b,2d,d)$

$=(-b,b,0,0)+(0,0,2d,d)$

$=b(-1,1,0,0)+d(0,0,2,1)$

$=$ L·c of elements of S_1

$S_1=\{(-1,1,0,0),(0,0,2,1)\}$

$\therefore \dim Q=2$

(iii) Now

$P\cap Q=\{(a,b,c,d) / b+c+d=0, a+b=0, c=2d\}$

Let us take

$\gamma = (a, b, c, d) \in P \cap Q$

$\Rightarrow b+c+d=0, \; a+b=0, \; c=2d$

$\Rightarrow -a+c+d=0 \quad \Rightarrow a=-b$

$\Rightarrow a=c+d \quad \Rightarrow b=-a$

$\Rightarrow a=3d \quad \Rightarrow b=-3d$

Now $\gamma = (a, b, c, d) = (3d, -3d, 2d, d)$

$= d(3, -3, 2, 1)$

$=$ L.c of one vector $(3, -3, 2, 1)$

$\therefore \dim(P \cap Q) = 1$